Mathematical Economics 102

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Fundamentals

refer to textbook

Ch.2 Economic Models

Ch.3 Equilibrium Analysis in Economics

Others: p.82-84, p.230-231, p.318-320, p.327-330,

$P \Rightarrow Q$ ($\equiv$ not $Q \Rightarrow$ not P) can be read as

- if P then Q
- P implies Q
- P only if Q
- P is a **sufficient** condition for Q
- Q is a **necessary** condition for P

ex: **P**: George is Mary's father.

Q: George is a male.

ex: **P**: All students in this class are undergraduates.

Q: No one in this class is under 10 years old.

ex: Prove that $\sqrt{2}$ is an irrational number.

ex: If you believe in me with all your heart,
you will be able to walk through that wall.

$P \Leftrightarrow Q$ (i.e. $P \Rightarrow Q$ and $Q \Rightarrow P$) can be read as

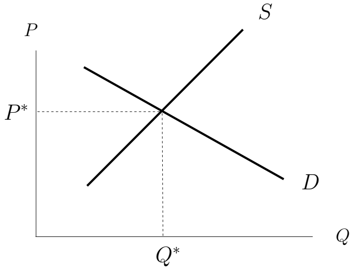
- P if and only if Q
- P is equivalent to Q
- P is a **necessary and sufficient** condition for Q
- P implies and is implied by Q

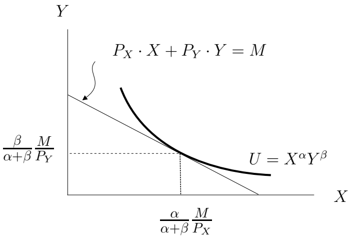
- A **variable** is something whose magnitude can change.
- A **constant** is a magnitude that does not change.
- A **parameter** is a constant that is variable.

ex: $Q_X^D = 25 - 2P_X + P_Y + 0.2M$ (a demand function)

$U(X, Y) = X^a Y^b$ (an utility function)

- **Endogenous** variables are those whose solution values we seek from the model.
- **Exogenous** variables are determined by forces external to the model and whose magnitudes are accepted as given data only.



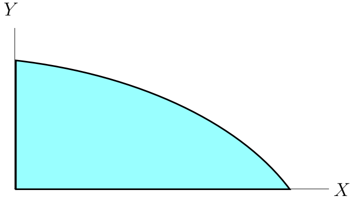


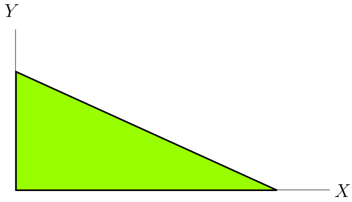
- A **definitional equation** sets up an identity between two alternate expressions that have exactly the same meaning.
  ex: $\pi \equiv R - C, \quad x^n \equiv x \times x \times \cdots \times x (n \text{ terms})$
- A **behavioral equation** specifies the manner in which a variable behaves in response to changes in other variables.
  ex: $C = Q^2 + 2Q + 35, \quad Y = K^{0.3} L^{0.7}$
- A **conditional equation** states a requirement to be satisfied.
  ex: $Q_d = Q_s, \quad I = S$

Sets

- A **set** is a collection of distinct items thought of as a whole, and these items are called the **elements** of the set.

ex: production possibility set budget set





Two ways of writing a set:

- **Enumeration**

ex: $A = \{1, 2, 3, 4\} = \{2, 4, 3, 1\}$

$\Rightarrow\ 3 \in A, \quad 5 \notin A$

$\mathbb{Z}_+ = \{1, 2, 3, 4, \ldots\}$

- **Description**

ex: $B = \{x | x \leq 4, x \in \mathbb{Z}_+\} = \{x \in \mathbb{Z}_+ : x \leq 4\}$

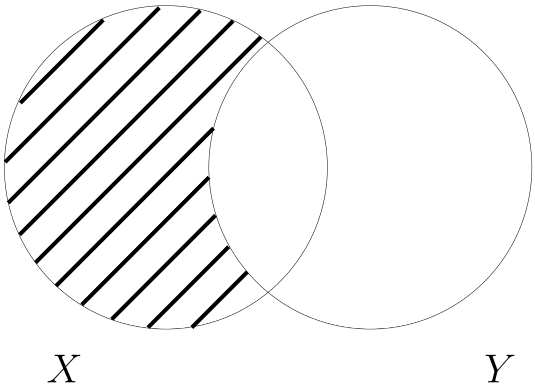
- X is a **subset** of Y if and only if all the elements of set X are also elements of set Y, and we write

$$X \subseteq Y$$

  where $\subseteq$ is the set-inclusion relation.

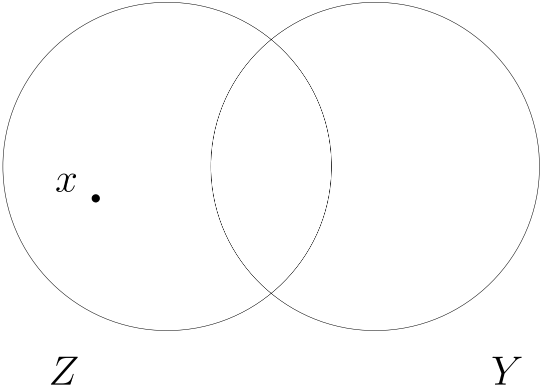
- Z is **not** a subset of Y iff there exists at least one x such that $x \in Z$ but $x \notin Y$ and we write

$$Z \nsubseteq Y$$



Venn Diagram

Note that there are no elements in the area filled by slanted lines.


x
Z
Y

- The **empty set** (or the **null set**) is the set with no elements. The empty set is always written ϕ or $\{ \}$.
- ϕ is a subset of **any** set.

proof:

If $\phi \nsubseteq A$, then there must be at least one element x such that $x \in \phi$ but $x \notin A$. However, there is **no** element in ϕ by definintion. Therefore, $\phi \subseteq A$.

- If there are m elements in set A, then there are 2^m subsets contained in set A.

ex: $A = \{1, 2, 3\}$

subsets of A:$\phi, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}$

- The **power set** of a set X is the set of all subsets of X, and is written $\mathcal{P}(X)$. That is, $\mathcal{P}(X) = \{A : A \subseteq X\}$.

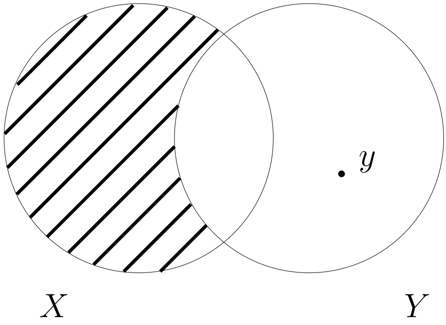
ex: $A = \{1\}$

$\mathcal{P}(A) = \{\phi, \{1\}\}$

$\mathcal{P}(\mathcal{P}(A)) = \{\phi, \{\phi\}, \{\{1\}\}, \{\phi, \{1\}\}\}$

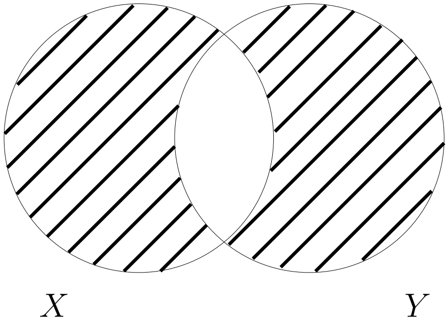
- X is a **proper subset** of Y iff all the elements in set X are in set Y, but not all the elements of Y are in X, and we write

$$X \subset Y \quad \text{iff} \quad X \subseteq Y \quad \text{but} \quad Y \nsubseteq X$$



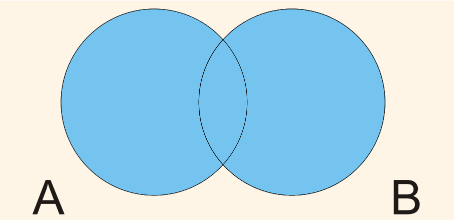
- Two sets X and Y are **equal** iff they contain exactly the same elements, and we write

$$X = Y \quad \text{iff} \quad X \subseteq Y \quad \text{and} \quad Y \subseteq X$$



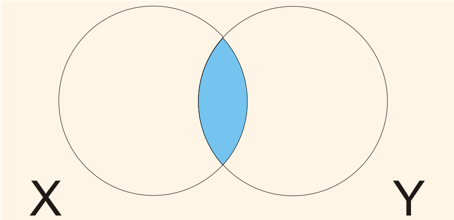
- The **union** of two sets A and B is the set of elements in one or other of the sets. We write

$$A \cup B = \{x : x \in A \textbf{ or } x \in B\}$$



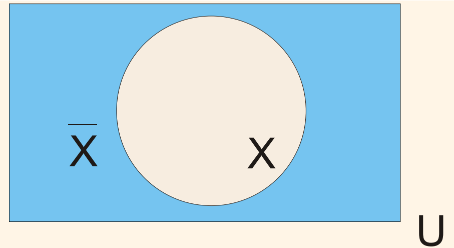
- The **intersection** of two sets X and Y is the set of elements that are in $both$ X and Y. We write

$$X \cap Y = \{x : x \in X \textbf{ and } x \in Y\}$$



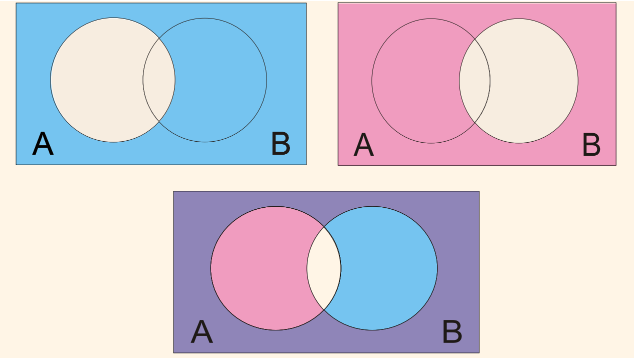
- The **complement** of a set X is the set of elements of the universal set U that are not elements of X, and it is written $\overline{X}$. Thus

$$\overline{X} = \{x \in U : x \notin X\}$$



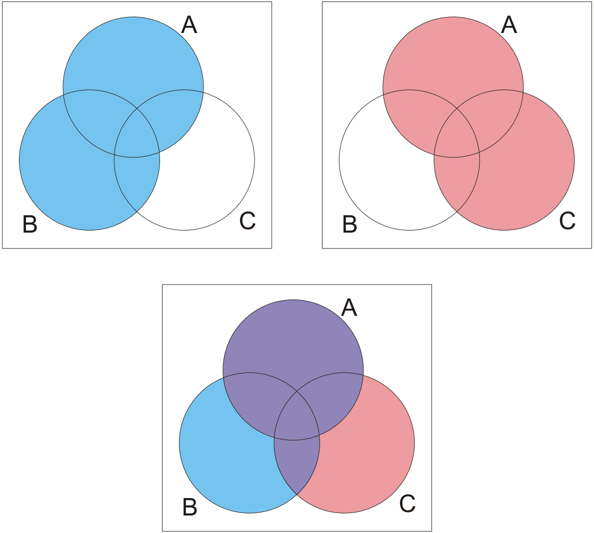
DeMorgan's Rule

$$(1)\overline{A \cup B} = \overline{A} \cap \overline{B} \qquad (2)\overline{A \cap B} = \overline{A} \cup \overline{B}$$



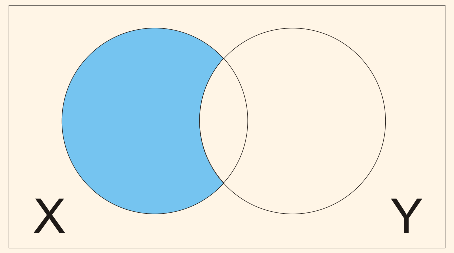
Laws of Set Operations

- **commutative law** $A \cup B = B \cup A$
  $A \cap B = B \cap A$
- **associative law** $A \cup (B \cup C) = (A \cup B) \cup C$
  $A \cap (B \cap C) = (A \cap B) \cap C$
  $A \cup (B \cap C) \neq (A \cup B) \cap C$
- **distributive law** $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$
  $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$


A
B
C
A
B
C
A
B
C

- The **relative difference** of X and Y, denoted $X - Y$, is the set of elements of X that are not also in Y

$$X - Y = \{x \in U : x \in X \text{ and } x \notin Y\}$$



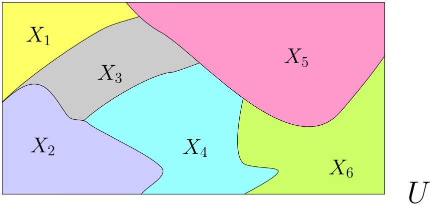
- A **partition** of the universal set U is a collection of disjoint subsets of U, the union of which is U. Thus, if we have n subsets X_i, $i = 1, \cdots, n$, such that

$$X_i \cap X_j = \phi, \quad i, j = 1, \cdots, n, \quad i \neq j$$

and

$$X_1 \cup X_2 \cup X_3 \cup \cdots \cup X_n = U$$

then these n subsets form a partition of U.



ex: Show that for any $X \subseteq U$, $\{X, \overline{X}\}$ is a partition of U.

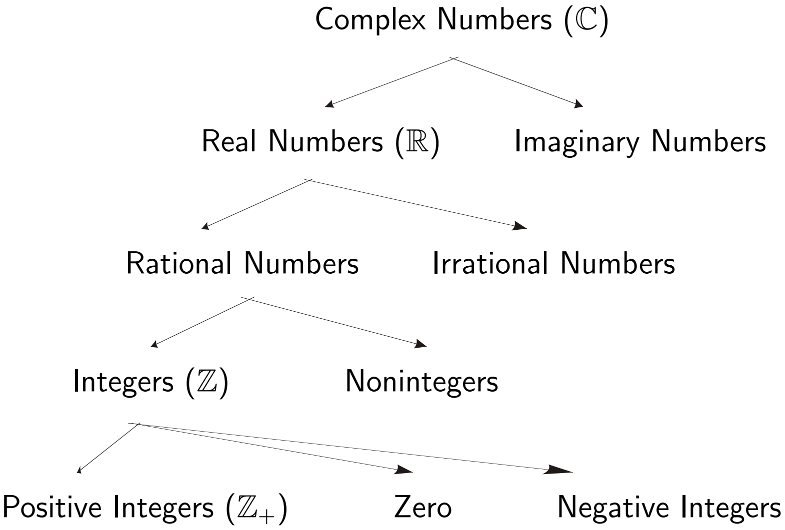
ex: Consider the collection of subsets of $\mathbb{Z}_+$ defined as follows:

$$X_i = \{x \in \mathbb{Z}_+ : 10(i-1) < x \leq 10i,\ i \in \mathbb{Z}_+\}$$

Does the collection of these X_i form a partition of $\mathbb{Z}_+$?

Solution

$$X_1 = \{x \in \mathbb{Z}_+ : 0 < x \leq 10\}$$
$$X_2 = \{x \in \mathbb{Z}_+ : 10 < x \leq 20\}$$
$$X_3 = \{x \in \mathbb{Z}_+ : 20 < x \leq 30\}$$


Complex Numbers ($\mathbb{C}$)
Real Numbers ($\mathbb{R}$)
Imaginary Numbers
Rational Numbers
Irrational Numbers
Integers ($\mathbb{Z}$)
Nonintegers
Positive Integers ($\mathbb{Z}_+$)
Zero
Negative Integers

- The set $\mathbb{R}_{++} \subset \mathbb{R}$ consists of the strictly positive real numbers with the characteristics that

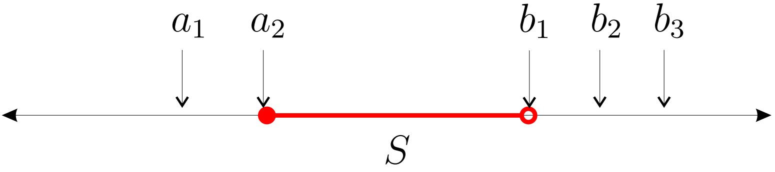
(i) $\mathbb{R}_{++}$ is closed under addition and multiplication.

(ii) For any $a \in \mathbb{R}$, exactly one of the following is true:
$a \in \mathbb{R}_{++}$ or $a = 0$ or $-a \in \mathbb{R}_{++}$

- The set $\mathbb{R}_{+} = \mathbb{R}_{++} \cup \{0\}$ is the set of **nonnegative** real numbers.

Bounded and Closed Sets

- A set $S \subset \mathbb{R}$ is **bounded above** if there *exists* $b \in \mathbb{R}$ such that for all $x \in S$, $x \leq b$; b is then called *an* upper bound of S.
- A set $S \subset \mathbb{R}$ is **bounded below** if there exists $a \in \mathbb{R}$ such that for all $x \in S$, $x \geq a$; a is then called a lower bound of S.



- The **supremum** of a set S, written **sup** S, has the properties:
  (i) $x \leq \sup S$ for all $x \in S$.
  (ii) If b is an upper bound of S, then $\sup S \leq b$.

- The **infimum** of a set S, written **inf** S, has the properties:
  (i) $x \geq \inf S$ for all $x \in S$.
  (ii) If a is a lower bound of S, then $a \leq \inf S$.

Conclusions

- If the sup or the inf of a subset of $\mathbb{R}$ exists, then it is unique.
- Every nonempty subset of $\mathbb{R}$ that has an upper bound has a supremum (least upper bound) in $\mathbb{R}$.
- Every nonempty subset of $\mathbb{R}$ that has a lower bound has an infimum (greatest lower bound) in $\mathbb{R}$.
- If sup $X \in X$, then sup X is called the maximum of X. In the same way, if inf $X \in X$, then inf X is called the minimum of X.

An interval is **bounded** if it is *impossible* to go off to infinity while remaining inside it.

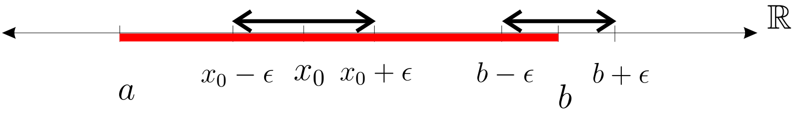
- unbounded above

$$[a, \infty) = \{x \in \mathbb{R} : x \geq a\}$$
$$(a, \infty) = \{x \in \mathbb{R} : x > a\}$$

- unbounded below

$$(-\infty, b] = \{x \in \mathbb{R} : x \leq b\}$$
$$(-\infty, b) = \{x \in \mathbb{R} : x < b\}$$

- A **boundary point** of an interval, such as $[a, b]$, is a point x_0 that *every* interval $(x_0 - \epsilon, x_0 + \epsilon)$ around it, *however small*, must contain points that are in $[a, b]$ and points that are not.
- For an **interior point** of $[a, b]$, it is *always possible* to find an interval $I_\epsilon(x_0)$ that lies *entirely* in $[a, b]$.



A **closed** interval contains all (*if any*) its **boundary points**.

- **closed interval** : $[a,b] = \{x \in \mathbb{R} : a \leq x \leq b\}$
- **half-open interval** : $[a,b) = \{x \in \mathbb{R} \ : a \leq x < b\}$

  $(a,b] = \{x \in \mathbb{R} : a < x \leq b\}$
- **open interval** : $(a,b) = \{x \in \mathbb{R} : a < x < b\}$

A **compact** interval is defined as an interval that is both **closed** and **bounded**.

ex: $[2, 5]$ closed and bounded

ex: $[2, 5)$ half-open and bounded

ex: $[2, \infty)$ closed and unbounded above

ex: $(-\infty, 5)$ open and unbounded below

Euclidean Space

- ordered pairs (a, b)

  Note: $(a, b) \neq (b, a)$ unless $a = b$

- ordered triples (a, b, c)
- ordered quadruple (a, b, c, d)
- ordered quintuple (a, b, c, d, e)

The **cartesian product** of two sets X and Y, written $X \otimes Y$, is the set of ordered pairs formed by taking in turn each element in X and associating with it each element in Y

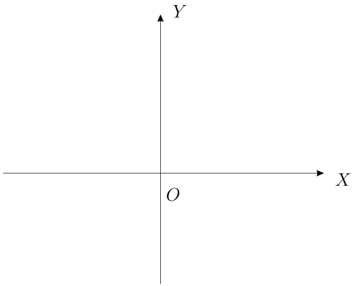
$$X \otimes Y \equiv \{(a, b) : a \in X \text{ and } b \in Y\}$$

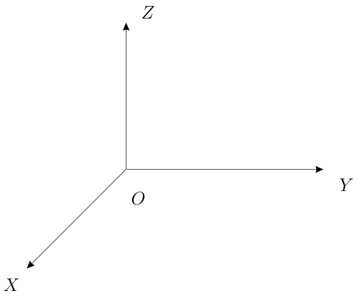
ex: $X = \{1, 2, 3\}, \quad Y = \{a, b\}$

$$X \otimes Y = \{(1, a), (1, b), (2, a), (2, b), (3, a), (3, b)\}$$

ex: $\mathbb{R} \otimes \mathbb{R} = \mathbb{R}^2 = \{(x, y) : x \in \mathbb{R}, y \in \mathbb{R}\}$

ex: $\mathbb{R} \otimes \mathbb{R} \otimes \mathbb{R} = \mathbb{R}^3 = \{(x, y, z) : x \in \mathbb{R}, y \in \mathbb{R}, z \in \mathbb{R}\}$


Y
X
O


Z
Y
O
X

Given points $a = (a_1, \ldots, a_N)$ and $b = (b_1, \ldots, b_N)$ in $\mathbb{R}^N$, $N \geq 1$, the **Euclidean distance** between them is

$$d(a,b) = \sqrt{\sum_{i=1}^{N} (a_i - b_i)^2}$$

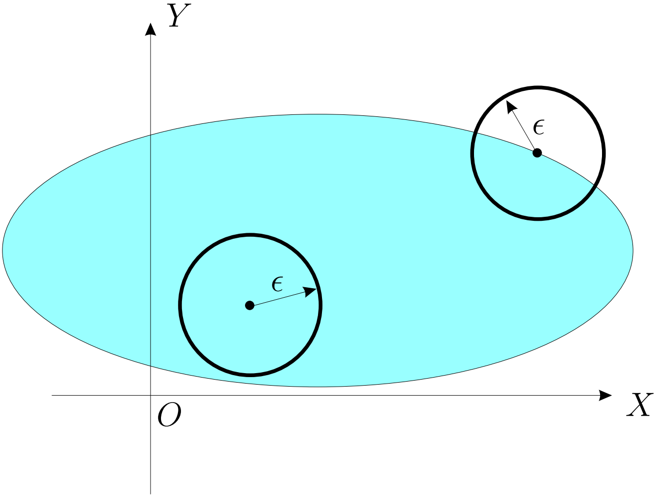
ex: $a = a_1, b = b_1$,

$$d(a,b) = \sqrt{(a_1 - b_1)^2} = |a_1 - b_1|$$

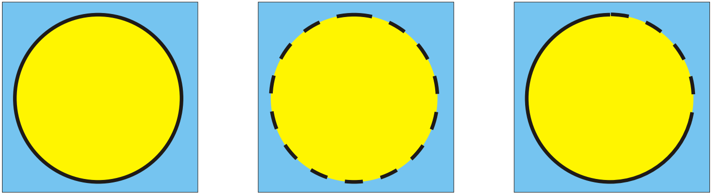
ex: $a = (a_1, a_2), b = (b_1, b_2)$,

$$d(a,b) = \sqrt{(a_1 - b_1)^2 + (a_2 - b_2)^2}$$

- An **ϵ-neighborhood** of a point $\mathbf{x}_0 \in \mathbb{R}^N$ is given by the set $N_\epsilon(\mathbf{x}_0) = \{\mathbf{x} \in \mathbb{R}^N : d(\mathbf{x}_0, \mathbf{x}) < \epsilon, \epsilon \in \mathbb{R}_{++}\}$. Simply, $N_\epsilon(\mathbf{x}_0)$ is the set of points lying *within* a distance ϵ of $\mathbf{x}_0$.
- A **boundary point** of a set $X \subset \mathbb{R}^N$ is a point $\mathbf{x}_0$ such that *every* ϵ-neighborhood $N_\epsilon(\mathbf{x}_0)$ contains points that are in and points that are not in X.
- An **interior point** of a set $X \subset \mathbb{R}^N$ is a point $\mathbf{x}_0 \in X$ for which there *exists* an ϵ such that $N_\epsilon(\mathbf{x}_0) \subset X$.


Y
ϵ
ϵ
O
X

- A set $X \subset \mathbb{R}^N$ is **open** if, for every $\mathbf{x} \in X$, there *exists* an ϵ such that $N_\epsilon(\mathbf{x}) \subset X$. That is, an open set is composed of its interior points only.
- A set $X \subset \mathbb{R}^N$ is **closed** if *all* the boundary points of X are also in the set X.



Note: Points in the broken part on the circumference of X (the yellow disk) do not belong to X, while points in the solid part do.

- The **interior** of a set $X \subset \mathbb{R}^N$ is the open set

  $$\text{Int } X = \{\mathbf{x} \in \mathbb{R}^N : \mathbf{x} \text{ is an interior point of } X\}$$

  (the disk without its circumference)

- The **closure** of X is the closed set

  $$\text{Cl } X = \mathbb{R}^N \setminus \text{Int}(\mathbb{R}^N \setminus X)$$

  (the disk with its entire circumference)

- The **boundary** of X is the closed set

  $$\begin{aligned}\text{Bdry } X &= \text{Cl } X \setminus \text{Int } X \\ &= \{\mathbf{x}' \in \mathbb{R}^N : \mathbf{x}' \text{ is a boundary point of } X\}\end{aligned}$$

  (the entire circumference only)

- A set $X \subset \mathbb{R}^N$ is **open iff** its complement $\overline{X} \subset \mathbb{R}^N$ is a **closed** set.

Proof

(i) Suppose that $\overline{X}$ is not a closed set, then at least one of its boundary points, say $\mathbf{x}$, is not in $\overline{X}$. That is, $\mathbf{x} \notin \overline{X}$ and thus $\mathbf{x} \in X$.

(ii) Because $\mathbf{x}$ is a boundary point of $\overline{X}$, every ϵ-neighborhood $N_\epsilon(\mathbf{x})$ contains points that are in and points that are not in $\overline{X}$. Hence, $\mathbf{x}$ is also a boundary point of X.

From (i) and (ii), X is not an open set.

- $\mathbb{R}^N \subseteq \mathbb{R}^N$ is both closed and open.

Proof

(i) For any point $\mathbf{x} \in \mathbb{R}^N$, we can find an $\epsilon > 0$ such that $N_\epsilon(\mathbf{x}) \subset \mathbb{R}^N$. Hence, all points in $\mathbb{R}^N$ are its interior points and then $\mathbb{R}^N$ is an open set.

(ii) Since all points in $\mathbb{R}^N$ are interior points, all (if any) its boundary points will be in its complement ϕ. However, $\phi \subset \mathbb{R}^N$ and then all its boundary points are also in $\mathbb{R}^N$. Thus, $\mathbb{R}^N$ is a closed set.

- ϕ is both closed and open.

- The intersection of two open sets is open.

Proof

Assume that $X, Y \subset \mathbb{R}^N$ are open and $Z = X \cap Y$.

(i) If $Z = \phi$, then it is an open set.

(ii) If $Z \neq \phi$, then for any $\mathbf{z}_0 \in Z$, we will have $\mathbf{z}_0 \in X$ and $\mathbf{z}_0 \in Y$. Since X and Y are open, there must exist $\epsilon_x > 0$ and $\epsilon_y > 0$ such that $N_{\epsilon_x}(\mathbf{z}_0) \subset X$ and $N_{\epsilon_y}(\mathbf{z}_0) \subset Y$. Let $\epsilon = \min\{\epsilon_x, \epsilon_y\}$, $N_\epsilon(\mathbf{z}_0) \subset X$ and $N_\epsilon(\mathbf{z}_0) \subset Y$ and thus $N_\epsilon(\mathbf{z}_0) \subset Z$ will hold.

From (i) and (ii), Z is an open set.

- The union of two closed sets is closed.

Proof:

Assume that $X, Y \subset \mathbb{R}^N$ are closed and $Z = X \cup Y$.

(i) $\overline{X}, \overline{Y}$ are open.

(ii) $\overline{Z} = \overline{X} \cap \overline{Y}$ is open.

(iii) Z is closed.

- The union of two open sets is open.

- The intersection of two closed sets is closed.

- A set $X \subset \mathbb{R}^N$ is **bounded** if, for every $\mathbf{x}_0 \in X$, there *exists* a finite $\epsilon < \infty$ such that $X \subset N_\epsilon(\mathbf{x}_0)$.

- The intersection of two bounded sets is bounded.

Proof

Assume that $X, Y \subset \mathbb{R}^N$ are bounded and $Z = X \cap Y$. For any $\mathbf{z}_0 \in Z$, we will have $\mathbf{z}_0 \in X$. Since X is bounded, there must exists $0 < \epsilon < \infty$ such that $Z \subseteq X \subset N_\epsilon(\mathbf{z}_0)$. Hence Z is bounded.

- The union of two bounded sets is bounded.

- Consider a parameterized maximization problem of the form

$$M(a) = \max f(\mathbf{x}, a) \quad \text{such that} \quad \mathbf{x} \in G(a).$$

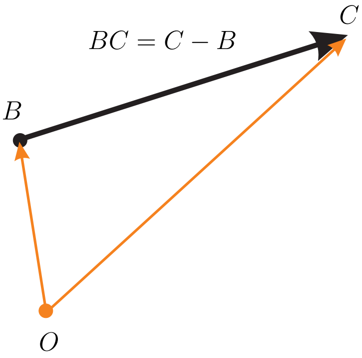
- **Existence of an optimum**

  If the constraint set $G(a)$ is *nonempty* and *compact*, and the function f is *continuous*, then there *exists* a solution $\mathbf{x}^*$ to this maximization problem.

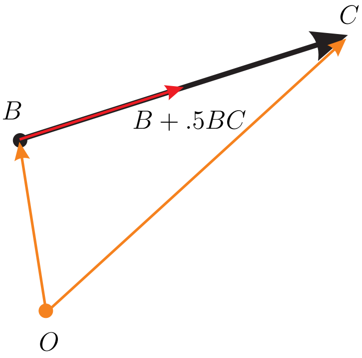
- **Uniqueness of optimum**

  If the function f is *strictly concave* and the constraint set is *convex*, then a solution, should it exist, is *unique*.

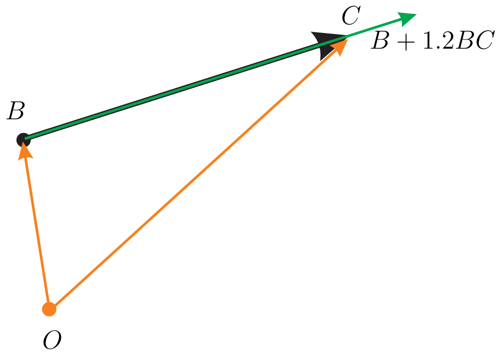
Convex Sets



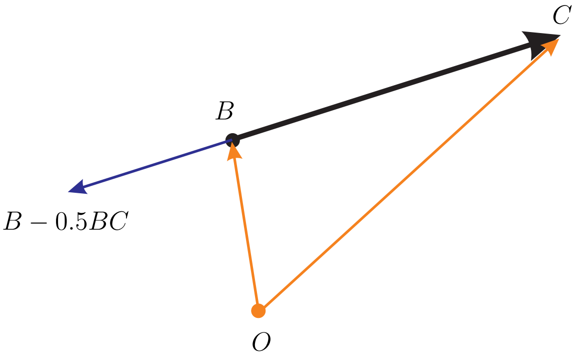
Convex Sets



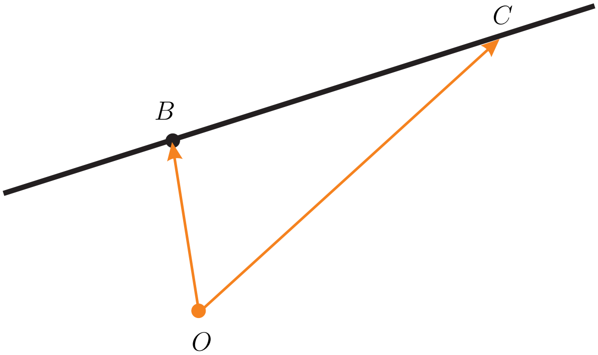
Convex Sets



Convex Sets



Convex Sets



$$\text{any point on } \overline{BC} = B + \lambda BC$$
$$= B + \lambda(C - B) = (1-\lambda)B + \lambda C$$

Convex Combination

- Given two points

$$\mathbf{x}' = (x'_1, x'_2, \cdots, x'_N)^T \in \mathbb{R}^N$$

and

$$\mathbf{x}'' = (x''_1, x''_2, \cdots, x''_N)^T \in \mathbb{R}^N,$$

their **convex combination** is the **set** of points $\overline{\mathbf{x}} \in \mathbb{R}^N$ for some $\lambda \in [0, 1]$, given by

$$\begin{aligned} \overline{\mathbf{x}} &= \lambda \mathbf{x}' + (1-\lambda)\mathbf{x}'' \\ &= [\ \lambda x'_1 + (1-\lambda)x''_1, \cdots, \lambda x'_N + (1-\lambda)x''_N\ ]^T \end{aligned}$$

- A set $X \subset \mathbb{R}^N$ is **convex** if for every pair of points $\mathbf{x}', \mathbf{x}'' \in X$, and any $\lambda \in [0, 1]$, the point

$$\overline{\mathbf{x}} = \lambda \mathbf{x}' + (1 - \lambda)\mathbf{x}''$$

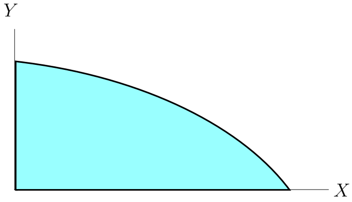
also belongs to the set X.

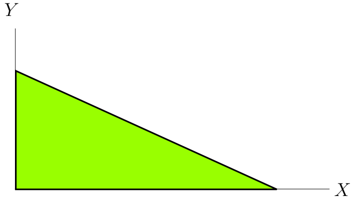
- A set $X \subset \mathbb{R}^N$ is **strictly convex**, if for every pair of points $\mathbf{x}', \mathbf{x}'' \in X$, $\mathbf{x}' \neq \mathbf{x}''$, and every $\lambda \in (0, 1)$, we have that $\overline{\mathbf{x}}$ is an interior point of X, where

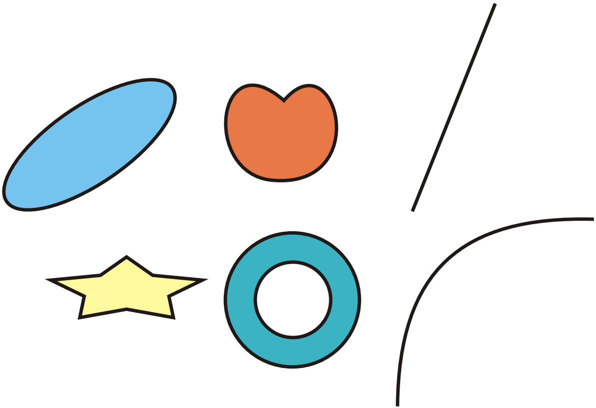
$$\overline{\mathbf{x}} = \lambda \mathbf{x}' + (1 - \lambda)\mathbf{x}''$$

- The intersection of two convex sets is also convex.
- The sum of two convex sets is also convex.

ex: production possibility set budget set





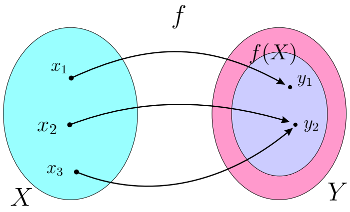


Functions

- Given two sets X and Y, a **function** (or a **mapping** / **transformation**) from X to Y is a rule that associates with each element of X, **one and only one** element of Y.

$$f : X \to Y \qquad \text{or} \qquad y = f(x), \quad x \in X$$

where x is referred to as the independent variable and y as the dependent variable.



- The set X is called the **domain** of the function, Y is called the **codomain**, and the set of elements in Y associated with the elements of X by the function is called the **range** of the function.
- The range of a function can be written as the **image set**.

$$f(X) = \{y \in Y : y = f(x),\ x \in X\}$$

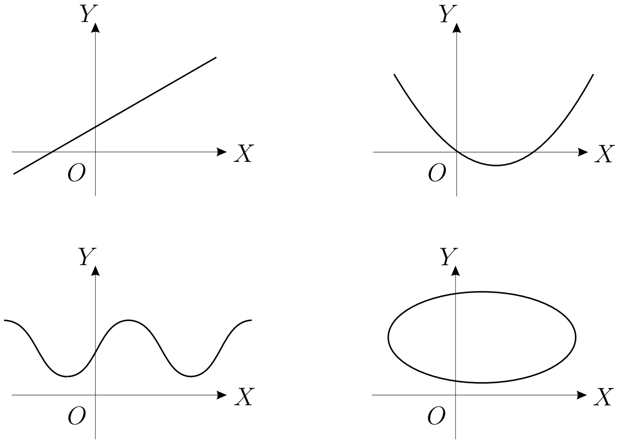
- If $f(X) \subset Y$, then we say f maps X into Y, while if $f(X) = Y$, then we say that f maps X onto Y.

- If we focus on cases in which $Y = \mathbb{R}$ and $X \subseteq \mathbb{R}^N, N \geq 1$, then $f : X \to Y$ will be referred to as a **real-valued** function.

ex: $y = f(x) = 2 + 3x, \quad x \in \mathbb{R}$

$$y = h(x_1, x_2) = {x_1}^2 {x_2}^3, \quad (x_1, x_2) \in \mathbb{R}^2_+$$

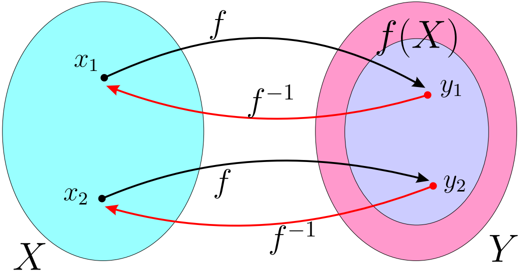
$$y = g(x, z, w) = \sin x + 2z - 3w^2, \quad (x, z, w) \in \mathbb{R}^3$$


Y
X
O
Y
X
O
Y
X
O
Y
X
O

- The **inverse function** of $y = f(x)$ is to invert this mapping and write x as a function of y, written as

$$x = f^{-1}(y)$$

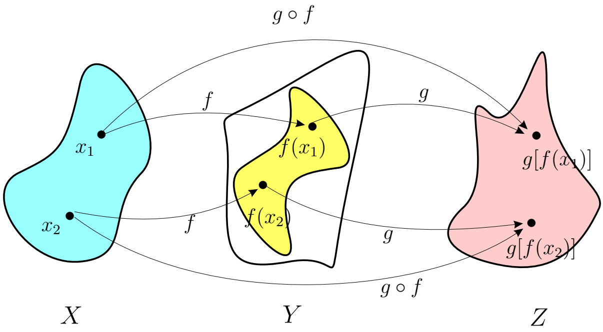
- This can only be done if f is **one-to-one** (into or onto).



- The **composite mapping** of two mappings $f : X \to Y$ and $g : Y \to Z$ is defined as

$$g \circ f : X \to Z \qquad \text{or} \qquad z = g[f(x)]$$

where $f(X) \subseteq Y$.



Types of Functions

- **Polynomial**

  $y = f(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0 x^0$

  ex: $y = 3$ (constant) ex: $y = 2x + 1$ (linear)

  ex: $y = x^2 + 2x + 5$ (quadratic) ex: $y = x^3 + 1$ (cubic)

- **Rational** $=$ a ratio of two polynomials in x

  ex: $y = \dfrac{x - 1}{x^2 + 2x + 4}$

- **Algebraic** = functions expressed in terms of polynomials and/or roots of polynomials
  ex: $y = \sqrt{x^2 + 3}$

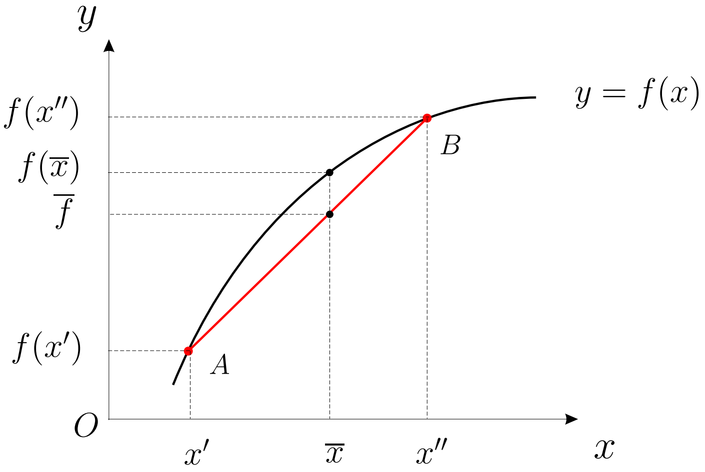
- **Nonalgebraic(Transcedential)**
  ex: $y = 3^x$ (exponential)
  ex: $y = \log_2 x$ (logarithmic)
  ex: $y = \sin x$ (trigonometric)

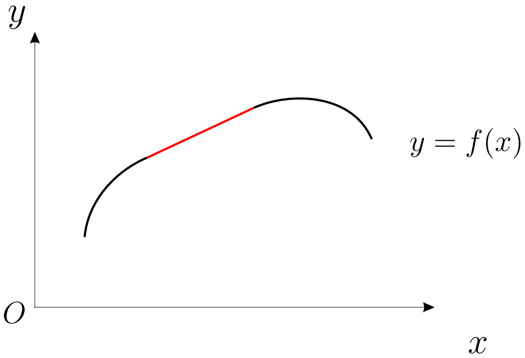
Concave and Quasiconcave functions

- Let $X \subset \mathbb{R}^N$ be a **convex** set and $f : X \to \mathbb{R}$. If for any two points $\mathbf{x}', \mathbf{x}'' \in X$ and $\lambda \in [0, 1]$,

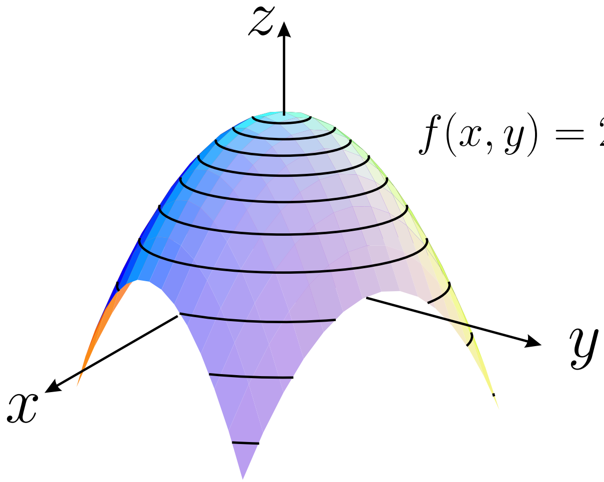
$$f(\overline{\mathbf{x}}) \geq \lambda f(\mathbf{x}') + (1 - \lambda) f(\mathbf{x}'') = \overline{f}$$

where $\overline{\mathbf{x}} = \lambda \mathbf{x}' + (1 - \lambda)\mathbf{x}''$, then f is said to be a **concave** function. That is, the line segment connecting points A and B lies **on or below** the surface.


y
$y = f(x)$
$f(x'')$
$f(\overline{x})$
$\overline{f}$
$f(x')$
B
A
O
x'
$\overline{x}$
x''
x



- The function f is **strictly concave** if the strict inequality holds when $\mathbf{x}' \neq \mathbf{x}''$ and $\lambda \in (0, 1)$, i.e., $\overline{AB}$ lies **entirely below** the surface except for A and B.

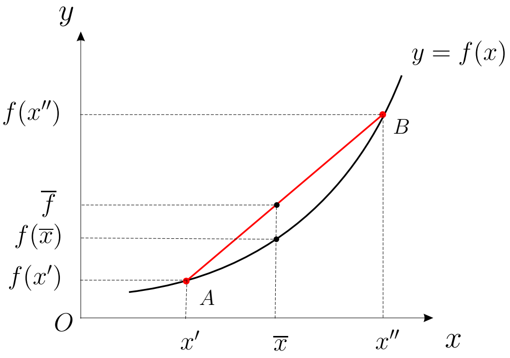


$$f(x, y) = 2 - x^2 - y^2$$

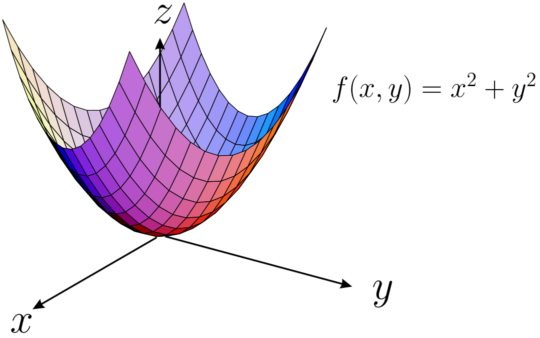
- The function f is **convex** if

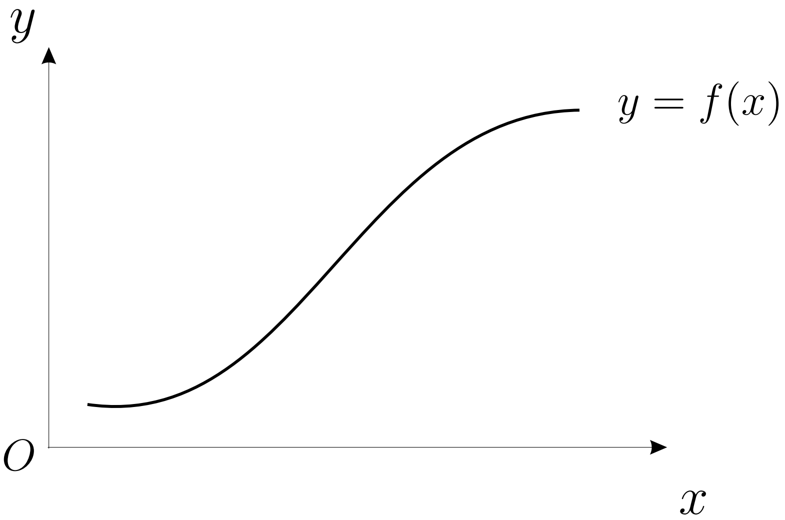
$$f(\overline{\mathbf{x}}) \leq \lambda f(\mathbf{x}') + (1-\lambda) f(\mathbf{x}'') = \overline{f}$$

where $\overline{\mathbf{x}} = \lambda \mathbf{x}' + (1-\lambda)\mathbf{x}''$ and $\lambda \in [0,1]$. That is, the line segment connecting points A and B lies **on or above** the surface.



- The function f is **strictly convex** if the strict inequality holds when $\mathbf{x}' \neq \mathbf{x}''$ and $\lambda \in (0, 1)$, i.e., $\overline{AB}$ lies **entirely above** the surface except for A and B.




y
$y = f(x)$
O
x

- $X \subset \mathbb{R}^N$, suppose that $f : X \to \mathbb{R}$ and $g : X \to \mathbb{R}$ are two **concave** functions. Show that $f + g$ is **concave**.

Proof

Let $\mathbf{x}'$, $\mathbf{x}'' \in X$, $\overline{\mathbf{x}} = \lambda\mathbf{x}' + (1-\lambda)\mathbf{x}''$ and $\lambda \in [0,1]$. Because

$$
\begin{aligned}
h(\overline{\mathbf{x}}) = f(\overline{\mathbf{x}}) + g(\overline{\mathbf{x}}) \\
&\geq [\lambda f(\mathbf{x}') + (1-\lambda) f(\mathbf{x}'')] + [\lambda g(\mathbf{x}') + (1-\lambda) g(\mathbf{x}'')] \\
&= \lambda[f(\mathbf{x}') + g(\mathbf{x}')] + (1-\lambda)[f(\mathbf{x}'') + g(\mathbf{x}'')] \\
&= \lambda h(\mathbf{x}') + (1-\lambda) h(\mathbf{x}''),
\end{aligned}
$$

then the sum of two concave functions is also concave.

- $X \subset \mathbb{R}^N$, if $f : X \to \mathbb{R}$ and $g : X \to \mathbb{R}$ are two **concave** functions and at least one of them is **strictly concave**, then $f + g$ is **strictly concave**.
- The sum of two **convex** functions is also **convex**. And if at least one of them is **strictly convex**, their sum will be **strictly convex**.
- The **negative** of a (strictly) concave function is (strictly) convex.

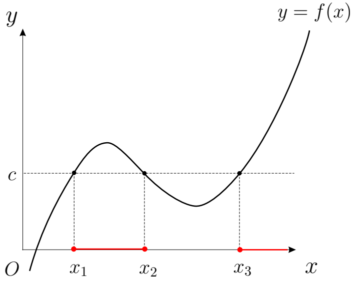
- A **level set** of the function $y = f(\mathbf{x})$ is the set

$$L = \{\mathbf{x} \in \mathbb{R}^N : f(\mathbf{x}) = c\}$$

for some given number $c \in \mathbb{R}$

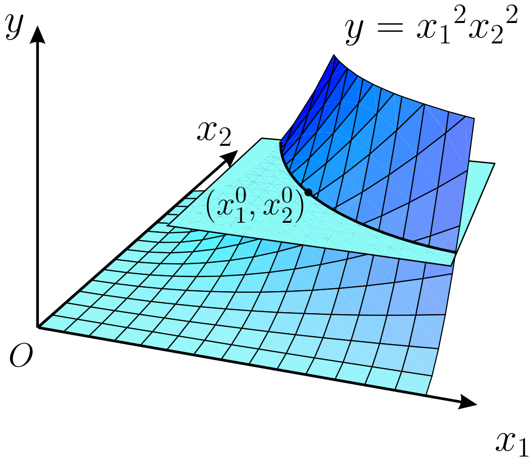
- The **better set** of the point $\mathbf{x}_0$ is

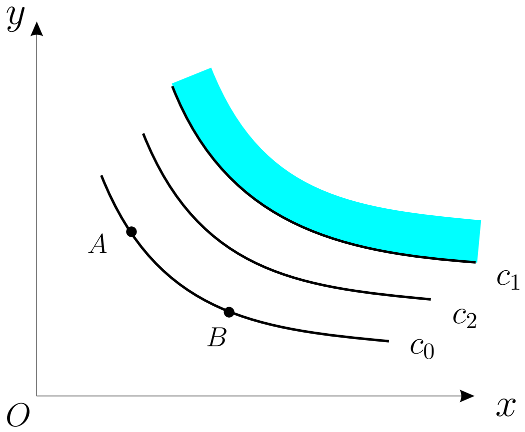
$$B(\mathbf{x}_0) = \{\mathbf{x} : f(\mathbf{x}) \geq f(\mathbf{x}_0)\}$$



- $L(c) = \{x_1, x_2, x_3\}$

- $B(x_1) = B(x_2) = B(x_3)$
$= \{x \in \mathbb{R} : x \in [x_1, x_2] \cup [x_3, \infty)\}$


y
$y = {x_1}^2 {x_2}^2$
x_2
(x_1^0, x_2^0)
O
x_1


y
A
B
c_1
c_2
c_0
O
x

- $X \subset \mathbb{R}^N$, suppose that $f : X \to \mathbb{R}$ is a **concave** function. Show that, for every point $\mathbf{x}_0 \in X$, the better set $B(\mathbf{x}_0)$ is **convex**.

Proof

Let $\mathbf{x}'$, $\mathbf{x}'' \in B(\mathbf{x}_0)$, then $f(\mathbf{x}') \geq f(\mathbf{x}_0)$ and $f(\mathbf{x}'') \geq f(\mathbf{x}_0)$. Since f is a concave function, for any $\lambda \in [0, 1]$,

$$\begin{aligned} f(\overline{\mathbf{x}}) &\geq \lambda f(\mathbf{x}') + (1-\lambda) f(\mathbf{x}'') \\ &\geq \lambda f(\mathbf{x}_0) + (1-\lambda) f(\mathbf{x}_0) = f(\mathbf{x}_0). \end{aligned}$$

Thus, $\overline{\mathbf{x}} \in B(\mathbf{x}_0)$. That is, $B(\mathbf{x}_0)$ is a convex set.

- The better set is also called the **upper contour set**.
- The **worse set** (or the **lower contour set**) of the point $\mathbf{x}_0$ is

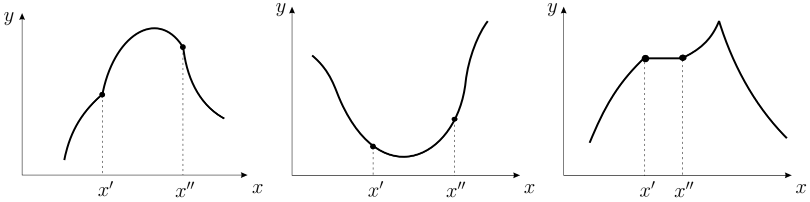
$$W(\mathbf{x}_0) = \{\mathbf{x} : f(\mathbf{x}) \leq f(\mathbf{x}_0)\}$$

- If $X \subset \mathbb{R}^N$, and $f : X \to \mathbb{R}$ is a convex function, then, for every point $\mathbf{x}_0 \in X$, the worse set $W(\mathbf{x}_0)$ is convex.

- f is (strictly) **quasiconcave** if and only if

$$f(\overline{\mathbf{x}}) \geq (>) \text{Min}\{f(\mathbf{x}'), f(\mathbf{x}'')\}$$

for all $\mathbf{x}', \mathbf{x}'' \in X$ and $\lambda \in [0, 1]$. $(\mathbf{x}' \neq \mathbf{x}'', \quad \lambda \in (0, 1))$



- f is (strictly) **quasiconvex** if and only if

$$f(\overline{\mathbf{x}}) \leq (<) \text{Max}\{f(\mathbf{x}'), f(\mathbf{x}'')\}$$

for all $\mathbf{x}', \mathbf{x}'' \in X$ and $\lambda \in [0, 1]$. $(\mathbf{x}' \neq \mathbf{x}'', \quad \lambda \in (0, 1))$

- Let $X \subset \mathbb{R}^N$ be a convex set, $f : X \to \mathbb{R}$. Show that f is a **quasiconcave** function **iff**, for every point $\mathbf{x}_0 \in X$, the better set $B(\mathbf{x}_0)$ is **convex**.

  That is,

$$\mathbf{x}' \in B(\mathbf{x}_0) \text{ and } \mathbf{x}'' \in B(\mathbf{x}_0) \quad \Rightarrow \quad \overline{\mathbf{x}} \in B(\mathbf{x}_0)$$

  or

$$f(\mathbf{x}') \geq f(\mathbf{x}_0) \text{ and } f(\mathbf{x}'') \geq f(\mathbf{x}_0) \quad \Rightarrow \quad f(\overline{\mathbf{x}}) \geq f(\mathbf{x}_0)$$

  for any $\mathbf{x}', \mathbf{x}'' \in X$ and $\lambda \in [0, 1]$.

Proof

(i) If f is quasiconcave, then $B(\mathbf{x}_0)$ is convex.

Given $\mathbf{x}'$, $\mathbf{x}'' \in B(\mathbf{x}_0)$ so that $f(\mathbf{x}') \geq f(\mathbf{x}_0)$ and $f(\mathbf{x}'') \geq f(\mathbf{x}_0)$, since f is quasiconcave, for any $\lambda \in [0, 1]$,

$$f(\overline{\mathbf{x}}) \geq \text{Min}\{f(\mathbf{x}'), f(\mathbf{x}'')\} \geq f(\mathbf{x}_0)$$

$\Rightarrow \ \overline{\mathbf{x}} \in B(\mathbf{x}_0)$. That is, $B(\mathbf{x}_0)$ is convex.

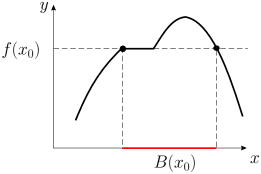
(ii) If $B(\mathbf{x}_0)$ is convex, then f is quasiconcave.

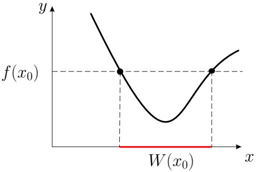
Assume that $f(\mathbf{x}') \geq f(\mathbf{x}'')$ so that $\mathbf{x}'$, $\mathbf{x}'' \in B(\mathbf{x}'')$.

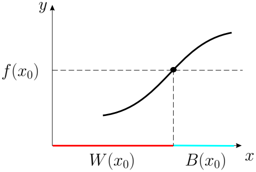
Since $B(\mathbf{x}'')$ is a convex set, $\overline{\mathbf{x}} \in B(\mathbf{x}'')$

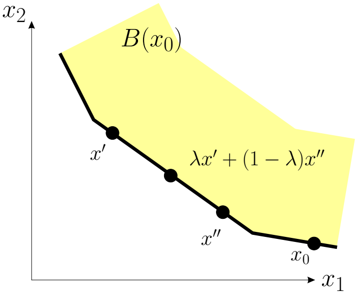
$\Rightarrow \ f(\overline{\mathbf{x}}) \geq f(\mathbf{x}'') = \text{Min}\{f(\mathbf{x}'), f(\mathbf{x}'')\}$

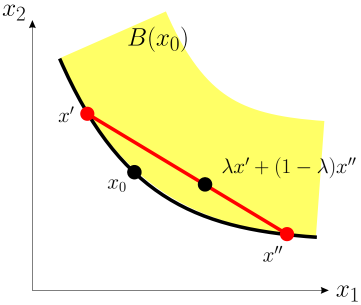
$\Rightarrow \ f$ is quasiconcave.


y
$f(x_0)$
$B(x_0)$
x


y
$f(x_0)$
$W(x_0)$
x


y
$f(x_0)$
$W(x_0)$
$B(x_0)$
x


x_2
$B(x_0)$
x'
$\lambda x' + (1-\lambda)x''$
x''
x_0
x_1

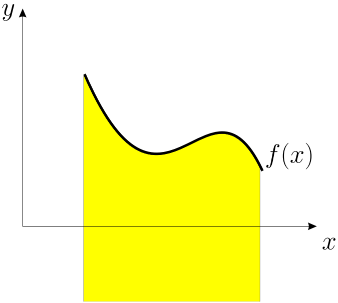

x_2
$B(x_0)$
x'
x_0
$\lambda x' + (1-\lambda)x''$
x''
x_1

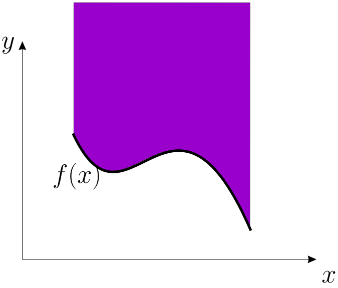
- $X \subset \mathbb{R}^N$, $f : X \to \mathbb{R}$, then the **hypograph** of f is a set defined as

$$HG_f = \{(\mathbf{x}, y) : \mathbf{x} \in X, y \in \mathbb{R}, y \leq f(\mathbf{x})\}$$

and the **epigraph** as

$$EG_f = \{(\mathbf{x}, y) : \mathbf{x} \in X, y \in \mathbb{R}, y \geq f(\mathbf{x})\}.$$





- If and only if f is a concave function, its hypograph is convex.

Proof:

(i) By definition, $(\mathbf{x}', f(\mathbf{x}')) \in HG_f$ and $(\mathbf{x}'', f(\mathbf{x}'')) \in HG_f$. Therefore, for $\lambda \in [0,1]$, $(\overline{\mathbf{x}}, \overline{f}) \in HG_f$ since HG_f is convex.

$\Rightarrow$ $\overline{f} \leq f(\overline{\mathbf{x}})$ by definition of HG_f. Thus, f is a concave function.

(ii) Assume that $(\mathbf{x}', y')$ and $(\mathbf{x}'', y'') \in HG_f$, thus $y' \leq f(\mathbf{x}')$ and $y'' \leq f(\mathbf{x}'')$. $\Rightarrow$ $\overline{y} \leq \overline{f} \leq f(\overline{\mathbf{x}})$

$\uparrow$

concave function

$\Rightarrow$ $(\overline{\mathbf{x}}, \overline{y}) \in HG_f$

$\Rightarrow$ HG_f is a convex set.

Linear Algebra

refer to textbook

Ch.4 Linear Models and Matrix Algebra

Ch.5 Linear Models and Matrix Algebra (continued)

- A **matrix** is a rectangular array of numbers enclosed in parentheses. It is conventionally denoted by a capital letter.

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 5 & 3 & 10 & 12 \\ 6 & 5 & 9 & 15 \\ 7 & 5 & 8 & 14 \\ 17 & 13 & 22 & 31 \\ 32 & 17 & 35 & 44 \end{bmatrix}$$

$$2 \times 2 \qquad\qquad 5 \times 4$$

- The number of rows and the number of columns determine the **dimension** (the **order**) of the matrix.

- A matrix A of order $m \times n$ can be explicitly written as

$$A = [\, a_{ij} \,], \quad i = 1 \sim m,\ j = 1 \sim n$$

$$= \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}_{m \times n}$$

- An array that consists of only one row or one column is known as a **vector**.

ex: $\begin{bmatrix} 5 & 3 & 5 & 4 \end{bmatrix}_{1\times 4}$ row matrix (row vector)

ex: $\begin{bmatrix} -1 \\ 2 \end{bmatrix}_{2\times 1}$ column matrix (column vector)

- Two matrices (say $A = [\ a_{ij}\ ],\ B = [\ b_{ij}\ ]$) are **equal** ($A = B$) iff (i) they have the same dimension and
(ii) all the corresponding elements are equal ($a_{ij} = b_{ij},\ \forall\ i, j$).

ex: $$\begin{bmatrix} 3 & 2 \\ x+y & 1 \end{bmatrix}_{2\times 2} = \begin{bmatrix} 3 & y \\ 2 & 1 \end{bmatrix}_{2\times 2}$$

$$\Rightarrow y = 2,\ x = 0.$$

ex: $$\begin{bmatrix} 3 & 4 & x \\ 2 & 5 & 7 \end{bmatrix}_{2\times 3} = \begin{bmatrix} 3 & w & 1 \\ z & 5 & y \end{bmatrix}_{2\times 3}$$

$$\Rightarrow x = 1,\ y = 7,\ z = 2,\ w = 4.$$

- A matrix that has the same number of rows and columns is called a **square matrix.**

ex: $A = \begin{bmatrix} 1 & 2 \\ 3 & 1 \end{bmatrix}_{2\times 2}$ **O** $B = \begin{bmatrix} 3 & 4 & 1 \\ 2 & 5 & 7 \end{bmatrix}_{2\times 3}$ **X**

$$C = \begin{bmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{bmatrix}_{3\times 3}$$ **O**

- Any square matrix that has only nonzero entries on the main diagonal and zeros everywhere else is known as a **diagonal matrix.**

ex: $P = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$, $Q = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 5 \end{bmatrix}$

$I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ **(identity matrix)**

- A matrix with all its entries being zero is known as the **null matrix.**

ex:

$$\mathbf{0}_{2\times 3} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\mathbf{0}_{3\times 3} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

- The **transpose matrix**, A^T(or A'), is the original matrix A with its rows and columns interchanged.

ex:

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 7 \end{bmatrix}_{2\times 3} \qquad A^T = \begin{bmatrix} 1 & 2 \\ 2 & 5 \\ 3 & 7 \end{bmatrix}_{3\times 2}$$

$$(A^T)^T = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 7 \end{bmatrix} = A$$

- A matrix A that is **equal** to its transpose A^T is called a **symmetric matrix**.

ex: $A = \begin{bmatrix} 5 & 1 \\ 9 & 3 \end{bmatrix}_{2\times 2}$ **X** $A^T = \begin{bmatrix} 5 & 9 \\ 1 & 3 \end{bmatrix}_{2\times 2}$

ex: $B = \begin{bmatrix} 1 & 2 & 4 \\ 3 & 8 & 0 \end{bmatrix}_{2\times 3}$ **X** $B^T = \begin{bmatrix} 1 & 3 \\ 2 & 8 \\ 4 & 0 \end{bmatrix}_{3\times 2}$

ex: $C = \begin{bmatrix} 1 & 3 & 5 \\ 3 & 2 & 8 \\ 5 & 8 & 4 \end{bmatrix}_{3\times 3}$ **O** $C^T = \begin{bmatrix} 1 & 3 & 5 \\ 3 & 2 & 8 \\ 5 & 8 & 4 \end{bmatrix}_{3\times 3}$

- The **sum** of two matrices is a matrix, the elements of which are the sums of the corresponding elements of the matrices.

$$[\ a_{ij}\ ] + [\ b_{ij}\ ] = [\ c_{ij}\ ], \quad \text{where } c_{ij} = a_{ij} + b_{ij},\ \forall\ i, j$$

ex: $\begin{bmatrix} 4 & 9 \\ 2 & 1 \end{bmatrix} + \begin{bmatrix} 2 & 0 \\ 0 & 7 \end{bmatrix} = \begin{bmatrix} 6 & 9 \\ 2 & 8 \end{bmatrix}$

ex: $\begin{bmatrix} 1 & 2 & 1 \\ 3 & 1 & 2 \end{bmatrix} + \begin{bmatrix} 0 & 3 & 5 \\ 1 & 2 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 5 & 6 \\ 4 & 3 & 1 \end{bmatrix}$

- Two matrices are **conformable** for addition if they have the same dimension. On the other hand, two matrices are **not conformable** for addition if their dimensions are different.

ex: $\begin{bmatrix} 1 & 2 \\ 3 & 1 \end{bmatrix} + \begin{bmatrix} 2 & -5 \\ 4 & 0 \end{bmatrix}$ **O**

$\begin{bmatrix} 3 & 4 & 1 \\ 6 & 5 & 3 \end{bmatrix} + \begin{bmatrix} 0 & 2 \\ -5 & 8 \end{bmatrix}$ **X**

- The sum of a matrix A and a (conformable) null matrix is A itself.

ex: $$\begin{bmatrix} 4 & 9 \\ 2 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 4 & 9 \\ 2 & 1 \end{bmatrix}$$

ex: $$\begin{bmatrix} 1 & 2 & 1 \\ 3 & 1 & 2 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 1 \\ 3 & 1 & 2 \end{bmatrix}$$

- The transpose of a sum of matrices is the sum of the transposes:

$$(A+B)^T = A^T + B^T$$

ex:
$$\begin{bmatrix} 4 & 9 \\ 2 & 1 \end{bmatrix}^T + \begin{bmatrix} 2 & 0 \\ 0 & 7 \end{bmatrix}^T = \begin{bmatrix} 4 & 2 \\ 9 & 1 \end{bmatrix} + \begin{bmatrix} 2 & 0 \\ 0 & 7 \end{bmatrix}$$
$$= \begin{bmatrix} 6 & 2 \\ 9 & 8 \end{bmatrix} = \begin{bmatrix} 6 & 9 \\ 2 & 8 \end{bmatrix}^T$$

- **Scalar multiplication** is carried out by multiplying each element of the matrix by the scalar.

$$k[\ a_{ij}\ ] = [\ ka_{ij}\ ] = [\ a_{ij}\ ]k$$

ex: $10 \begin{bmatrix} 1 & 3 \\ 5 & 7 \end{bmatrix} = \begin{bmatrix} 10 & 30 \\ 50 & 70 \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ 5 & 7 \end{bmatrix} 10$

ex: $2 \begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} -2 \\ 4 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \end{bmatrix} (-2)$

- Matrix subtraction can be defined by scalar multiplication and addition.

$$A - B = A + (-1)B$$

ex: $\begin{bmatrix} 1 & 2 \\ 3 & 1 \end{bmatrix} - \begin{bmatrix} 2 & -5 \\ 4 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 3 & 1 \end{bmatrix} + (-1)\begin{bmatrix} 2 & -5 \\ 4 & 0 \end{bmatrix}$

$= \begin{bmatrix} 1 & 2 \\ 3 & 1 \end{bmatrix} + \begin{bmatrix} -2 & 5 \\ -4 & 0 \end{bmatrix} = \begin{bmatrix} -1 & 7 \\ -1 & 1 \end{bmatrix}$

- Two matrices A and B of dimensions $m \times n$ and $n \times q$ respectively are **conformable** to form the product matrix

$$C_{m \times q} = A_{m \times n} B_{n \times q},$$

since the number of columns of the **lead matrix** A is equal to the number of rows of the **lag matrix** B.

- The ijth element of the product matrix, c_{ij}, is obtained by multiplying the elements of the ith row of A by the corresponding elements of the jth column of B and adding the resulting products.

- $[\ a_{ik}\ ]_{m\times n}\ [\ b_{kj}\ ]_{n\times q} = [\ c_{ij}\ ]_{m\times q}$, where $c_{ij} = \Sigma_k a_{ik} b_{kj},\ \forall i, j$

ex: $$\begin{bmatrix} 1 & 3 \\ 2 & 8 \\ 4 & 0 \end{bmatrix}_{3\times 2} \begin{bmatrix} 5 & 1 \\ 9 & 3 \end{bmatrix}_{2\times 2} = \begin{bmatrix} 1(5)+3(9) & 1(1)+3(3) \\ 2(5)+8(9) & 2(1)+8(3) \\ 4(5)+0(9) & 4(1)+0(3) \end{bmatrix}$$

$$= \begin{bmatrix} 32 & 10 \\ 82 & 26 \\ 20 & 4 \end{bmatrix}_{3\times 2}$$

ex: $$\begin{bmatrix} 2 & 4 \\ 1 & 2 \end{bmatrix}_{2\times 2} \begin{bmatrix} 1 & 2 & 2 \\ 3 & 1 & 4 \end{bmatrix}_{2\times 3} = \begin{bmatrix} 14 & 8 & 20 \\ 7 & 4 & 10 \end{bmatrix}_{2\times 3}$$

- The transpose matrix of the product matrix AB, where A and B are two conformable matrices, is defined as the product of the transposes, with the order of the multiplication reversed.

$$(AB)^T = B^T A^T$$

$$(ABC)^T = C^T (AB)^T = C^T B^T A^T$$

$$(ABCD)^T = D^T C^T B^T A^T$$

$$\left(\begin{bmatrix} 1 & 3 \\ 2 & 8 \\ 4 & 0 \end{bmatrix} \begin{bmatrix} 5 & 1 \\ 9 & 3 \end{bmatrix} \right)^T = \begin{bmatrix} 32 & 10 \\ 82 & 26 \\ 20 & 4 \end{bmatrix}^T$$

$$= \begin{bmatrix} 32 & 82 & 20 \\ 10 & 26 & 4 \end{bmatrix}$$

$$\begin{bmatrix} 5 & 1 \\ 9 & 3 \end{bmatrix}^T \begin{bmatrix} 1 & 3 \\ 2 & 8 \\ 4 & 0 \end{bmatrix}^T = \begin{bmatrix} 5 & 9 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 1 & 2 & 4 \\ 3 & 8 & 0 \end{bmatrix}$$

Q: $AB = BA$ **?**

A: In general, the product matrix AB (**premultiplying** B by A) does not equal the product matrix BA (**postmultiplying** B by A).

(i) AB or BA may not be well defined.

(ii) Even if both AB and BA are well defined, they are not equal in general.

ex: $A = \begin{bmatrix} 2 & 4 \\ 1 & 2 \end{bmatrix}_{2\times 2}$, $\qquad B = \begin{bmatrix} 1 & 2 & 2 \\ 3 & 1 & 4 \end{bmatrix}_{2\times 3}$

$AB = \begin{bmatrix} 14 & 8 & 20 \\ 7 & 4 & 10 \end{bmatrix}_{2\times 3}$, while BA is not well defined.

- Both of the product matrices AB and BA are well defined only if A and B are square matrices of the same order **or** for A of dimension $m \times n$ with B of dimension $n \times m$.

ex: $A = \begin{bmatrix} 2 & 4 \\ 1 & 3 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 2 \\ 2 & 0 \end{bmatrix}$

$$AB = \begin{bmatrix} 10 & 4 \\ 7 & 2 \end{bmatrix},\ BA = \begin{bmatrix} 4 & 10 \\ 4 & 8 \end{bmatrix} \quad \Rightarrow \mathbf{AB \neq BA}$$

ex: $A = \begin{bmatrix} 5 & 1 & 0 \\ 2 & 1 & -1 \end{bmatrix}$, $B = \begin{bmatrix} 4 & 3 \\ 1 & 1 \\ 0 & 2 \end{bmatrix}$

$$AB = \begin{bmatrix} 21 & 16 \\ 9 & 5 \end{bmatrix},\ BA = \begin{bmatrix} 26 & 7 & -3 \\ 7 & 2 & -1 \\ 4 & 2 & -2 \end{bmatrix} \quad \Rightarrow \mathbf{AB \neq BA}$$

- The multiplication of any matrix and a (conformable) null matrix is a null matrix.

ex: $$\begin{bmatrix} 2 & 4 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

- The multiplication of any matrix and a (conformable) identity matrix is the matrix itself.

ex: $$\begin{bmatrix} 2 & 4 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 4 \\ 1 & 2 \end{bmatrix}$$

Q: $AB = \mathbf{0} \quad \Rightarrow \quad A = \mathbf{0}$ or $B = \mathbf{0}$ **?**

ex: $$\begin{bmatrix} 2 & 4 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} -2 & 4 \\ 1 & -2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

A: Negative!

Q: $CD = CE \quad \Rightarrow \quad D = E$ **?**

ex: $$\begin{bmatrix} 2 & 3 \\ 6 & 9 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 5 & 8 \\ 15 & 24 \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ 6 & 9 \end{bmatrix} \begin{bmatrix} -2 & 1 \\ 3 & 2 \end{bmatrix}$$

A: Negative!

- The matrix A^n is the product matrix obtained by multiplying the **square** matrix A by itself n times.

- A square matrix A of any order is **idempotent** if

$$A = A^2 = A^3 = \cdots$$

where $A^2 = AA, A^3 = AAA$, etc.

ex: $A = \begin{bmatrix} 1/6 & -1/3 & 1/6 \\ -1/3 & 2/3 & -1/3 \\ 1/6 & -1/3 & 1/6 \end{bmatrix}$

- The **trace** of a square matrix A is given by the sum of the elements of the main diagonal. In other words, if A is $n \times n$, then the trace is defined as

$$\text{trace}(A_n) = a_{11} + a_{22} + \cdots + a_{nn}$$

ex: $A = \begin{bmatrix} 5 & 9 \\ 1 & 3 \end{bmatrix}$, $\text{trace}(A) = 8$

$B = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$, $\text{trace}(B) = 15$

- For two matrices A and B of dimensions $m \times n$ and $n \times m$ respectively, we have that AB is $m \times m$ and BA is $n \times n$ and

$$\text{trace}(AB) = \text{trace}(BA)$$

proof:

Let $C = AB$ and $D = BA$.

$$\begin{aligned}\text{trace}(AB) = \sum_{i=1}^{m} c_{ii} &= \sum_{i=1}^{m}\left(\sum_{j=1}^{n} a_{ij} b_{ji}\right) \\ &= \sum_{j=1}^{n}\left(\sum_{i=1}^{m} b_{ji} a_{ij}\right) = \sum_{j=1}^{n} d_{jj} = \text{trace}(BA)\end{aligned}$$

- The **inverse matrix** A^{-1} of a **square** matrix A of order n is the matrix that satisfies the condition that

$$AA^{-1} = A^{-1}A = I_n$$

where I_n is the identity matrix of order n.

- Any matrix A for which A^{-1} does not exist is known as a **singular matrix**.
- The matrix A for which A^{-1} exists is known as a **nonsingular matrix**.

Properties of the Inverse

- The inverse of an inverse matrix reproduces the original matrix

$$(A^{-1})^{-1} = A$$

- The inverse of a matrix is unique
- $(AB)^{-1} = B^{-1}A^{-1}$, provided that (i) A and B are of the same order, and (ii) A^{-1} and B^{-1} both exist.
- The inverse of the transpose equals the transpose of the inverse

$$(A^T)^{-1} = (A^{-1})^T$$

- The inverse of an inverse matrix reproduces the original matrix

$$(A^{-1})^{-1} = A$$

proof:

Let $B = (A^{-1})^{-1}$.

$\because A^{-1}B = A^{-1}(A^{-1})^{-1} = I$

$\Rightarrow AA^{-1}B = AI$

$\therefore B = A$ **Done!**

- The inverse of a matrix is unique

proof:

Assume that $AB = I$.

$\because A^{-1}AB = A^{-1}I$

$\Rightarrow B = A^{-1}$

$\therefore$ Any conformable matrix B satisfying $AB = I$ must be A^{-1}.

Done!

- $(AB)^{-1} = B^{-1}A^{-1}$, provided that (i) A and B are of the same order, and (ii) A^{-1} and B^{-1} both exist.

  proof:

  $$(AB)^{-1}(AB) = I$$
  $$\Rightarrow (AB)^{-1}ABB^{-1}A^{-1} = IB^{-1}A^{-1}$$
  $$\Rightarrow (AB)^{-1} = B^{-1}A^{-1}$$

Done!

- The inverse of the transpose equals the transpose of the inverse

$$(A^T)^{-1} = (A^{-1})^T$$

proof:

$$(A^{-1})^T A^T = (AA^{-1})^T = I^T = I$$

$$\Rightarrow (A^{-1})^T = (A^T)^{-1}$$

Done!

For a **system of linear equations**,

$$\begin{array}{rcrcrcr} x & + & 2y & - & 2z & = & 0 \\ -x & + & y & + & z & = & 5 \\ 4x & - & y & + & 2z & = & 13 \end{array}$$

there are three interesting questions:

- Does a solution exist?
- How many solutions are there?
- Is there an efficient algorithm that computes actual solutions?

Way 1: Substitution

$$\begin{array}{rcrcrcrl} x & + & 2y & - & 2z & = & 0 & \quad (1a) \\ -x & + & y & + & z & = & 5 & \quad (2a) \\ 4x & - & y & + & 2z & = & 13 & \quad (3a) \end{array}$$

$$\Rightarrow \qquad x = -2y + 2z \qquad (1b)$$

$$\Rightarrow \qquad \begin{array}{rcrcrl} 3y & - & z & = & 5 & \quad (2b) \\ -9y & + & 10z & = & 13 & \quad (3b) \end{array}$$

$$\Rightarrow \qquad z = 3y - 5 \qquad (2c)$$

$$\Rightarrow \qquad 21y = 63 \qquad (3c)$$

$$\Rightarrow \qquad y = 3, \quad z = 4, \quad x = 2$$

Way 2: Gaussian Elimination

$$\begin{array}{rcrcrcrl} 1x & + & 2y & - & 2z & = & 0 & (1a) \\ -x & + & y & + & z & = & 5 & (2a) \\ 4x & - & y & + & 2z & = & 13 & (3a) \end{array}$$

$\Rightarrow$

$$\begin{array}{rcrcrcrl} x & + & 2y & - & 2z & = & 0 & (1a) \\ & & 3y & - & z & = & 5 & (2b) \\ & - & 9y & + & 10z & = & 13 & (3b) \end{array}$$

$\Rightarrow$

$$\begin{array}{rcrcrcrl} x & + & 2y & - & 2z & = & 0 & (1a) \\ & & 3y & - & z & = & 5 & (2b) \\ & & & & 7z & = & 28 & (3c) \end{array}$$

$\Rightarrow \quad z = 4, \quad y = 3, \quad x = 2$ (**back substitution**)

Way 2′: Gauss-Jordan Elimination

$$\begin{aligned} x + 2y - 2z &= 0 \qquad (1a) \\ 3y - z &= 5 \qquad (2b) \\ z &= 4 \qquad (3d) \end{aligned}$$

$\Rightarrow$

$$\begin{aligned} x + 2y &= 8 \qquad (1b) \\ 3y &= 9 \qquad (2c) \\ z &= 4 \qquad (3d) \end{aligned}$$

$\Rightarrow$

$$\begin{aligned} x &= 2 \qquad (1c) \\ y &= 3 \qquad (2c) \\ z &= 4 \qquad (3d) \end{aligned}$$

$$\begin{array}{rcrcrcl} x & + & 2y & + & 3z & = & 1 \\ 3x & + & 2y & + & z & = & 1 \end{array}$$

$\begin{bmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{bmatrix}$ is called the **coefficient matrix** of the system

and $\left[\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 3 & 2 & 1 & 1 \end{array}\right]$ the **augmented matrix**.

- A row of a matrix is said to have k **leading zeros** if the first k elements of the row are all zeros and the $(k+1)$th element of the row is not zero.
- A matrix is in **row echelon form** if each row has more leading zeros than the row preceding it.
- The first nonzero entry in each row of a row echelon matrix is called a **pivot**.

$$\left[\begin{array}{ccc|c} 1 & 2 & -2 & 0 \\ 0 & 3 & -1 & 5 \\ 0 & 0 & 7 & 28 \end{array}\right]$$

- **Elementary row operations:**

1. interchange two rows of a matrix
2. multiply each element in a row by the same nonzero number
3. change a row by adding to it a multiple of another row

- A row echelon matrix in which (1) each pivot is a 1 and (2) each column containing a pivot contains no other nonzero entries is said to be in **reduced row echelon form**.

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 4 \end{array}\right]$$

$$\begin{array}{rcrcrcl} x & + & 2y & + & 3z & = & 1 \\ 3x & + & 2y & + & z & = & 1 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 3 & 2 & 1 & 1 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 0 & -4 & -8 & -2 \end{array}\right]$$

$$\Rightarrow \left[\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 0 & 1 & 2 & 0.5 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & 2 & 0.5 \end{array}\right]$$

$$\Rightarrow \begin{array}{rcl} x & = & z \\ y & = & 0.5 - 2z \end{array} \Rightarrow$$ **infinitely many solutions!**

$$\begin{array}{rcrcl} x & + & 3y & = & 1 \\ 3x & + & y & = & 1 \\ 2x & + & 3y & = & 1 \end{array} \quad \Rightarrow \quad \left[\begin{array}{cc|c} 1 & 3 & 1 \\ 3 & 1 & 1 \\ 2 & 3 & 1 \end{array}\right]$$

$$\Rightarrow \quad \left[\begin{array}{rr|r} 1 & 3 & 1 \\ 0 & -8 & -2 \\ 0 & -3 & -1 \end{array}\right] \quad \Rightarrow \quad \left[\begin{array}{rr|r} 1 & 3 & 1 \\ 0 & 1 & 0.25 \\ 0 & -3 & -1 \end{array}\right]$$

$$\Rightarrow \quad \left[\begin{array}{rr|r} 1 & 0 & 0.25 \\ 0 & 1 & 0.25 \\ 0 & 0 & -0.25 \end{array}\right] \quad \Rightarrow$$ **no solution!**

$$\begin{array}{rcrcr} x & + & 3y & = & 1 \\ 3x & + & y & = & 1 \\ 2x & + & 2y & = & 1 \end{array} \quad \Rightarrow \quad \left[\begin{array}{cc|c} 1 & 3 & 1 \\ 3 & 1 & 1 \\ 2 & 2 & 1 \end{array}\right]$$

$$\Rightarrow \left[\begin{array}{rr|r} 1 & 3 & 1 \\ 0 & -8 & -2 \\ 0 & -4 & -1 \end{array}\right] \quad \Rightarrow \quad \left[\begin{array}{rr|r} 1 & 3 & 1 \\ 0 & 1 & 0.25 \\ 0 & -4 & -1 \end{array}\right]$$

$$\Rightarrow \left[\begin{array}{cc|r} 1 & 0 & 0.25 \\ 0 & 1 & 0.25 \\ 0 & 0 & 0 \end{array}\right] \quad \Rightarrow$$ **exactly one solution!**

$$B = \left[\begin{array}{ccccccc|c} \mathbf{1} & w & w & 0 & 0 & w & 0 & d \\ 0 & 0 & 0 & \mathbf{1} & 0 & w & 0 & d \\ 0 & 0 & 0 & 0 & \mathbf{1} & w & 0 & d \\ 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{1} & d \end{array}\right]$$

where w, d may be either zero or nonzero.

- If the jth column of the row echelon matrix B contains a pivot, then x_j is called a **basic** variable.
- If the jth column of B does not contain a pivot, then we call x_j a **free** variable.

- The **rank** of a matrix is the number of nonzero rows in its row echelon form.

- Let A be the coefficient matrix and $\widehat{A}$ be the corresponding augmented matrix. Then

1. rank$(A) \leq$ rank$(\widehat{A})$
2. rank$(A) \leq$ number of rows of A
3. rank$(A) \leq \#$ col.s of A

- A system of linear equations with coefficient matrix A and augmented matrix $\widehat{A}$ has **a** solution **if and only if**

$$\text{rank}(A) = \text{rank}(\widehat{A})$$

- A system of linear equations must have either (1) no solution, (2) one solution, or (3) infinitely many solutions.

- **If** a system has exactly **one** solution, **then** A has at least as many rows(or equations) as columns(or unknowns).

$$\#\text{ rows of } A \geq \#\text{ col.s of } A$$

- **If** a system has more unknowns than equations, **then** it must have either no solution or infinitely many solutions.
- If a system in which all the elements in RHS are 0, then it is called **homogeneous** and must have at least one solution.
- A homogeneous system of linear equations which has more unknowns than equations must have infinitely many solutions.
- A system with A will have **a** solution for **every** RHS **if and only if**

$$\text{rank}(A) = \# \text{ rows of } A$$

- **If** a system has more equations than unknowns, **then** there **exists** an RHS such that the resulting system has **no** solution.
- Any system having A will have at most one solution for **every** RHS **if and only if**

$$\text{rank}(A) = \#\ \text{col.s of } A$$

- A system has exactly one solution for **every** RHS **if and only if**

$$\#\ \text{rows of } A = \#\ \text{col.s of } A = \text{rank}(A)$$

$$
\begin{array}{rcrcr}
5x & + & 2y & = & 3 \\
-x & - & 4y & = & 3
\end{array}
$$

$$
\Rightarrow \quad \begin{bmatrix} 5 & 2 \\ -1 & -4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \end{bmatrix}
$$

$$
\begin{array}{rcrcrcr}
4x & - & y & + & 2z & = & 13 \\
x & + & 2y & - & 2z & = & 0 \\
-x & + & y & + & z & = & 5
\end{array}
$$

$$
\Rightarrow \quad \begin{bmatrix} 4 & -1 & 2 \\ 1 & 2 & -2 \\ -1 & 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 13 \\ 0 \\ 5 \end{bmatrix}
$$

$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= d_1 \\
a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= d_2 \\
&\vdots \\
a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n &= d_n
\end{aligned}
$$

$$
\Rightarrow \quad \underbrace{\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}}_{n \times n} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}}_{n \times 1} = \underbrace{\begin{bmatrix} d_1 \\ d_2 \\ \vdots \\ d_n \end{bmatrix}}_{n \times 1}
$$

Quiz

Consider the linear system of equations $A\mathbf{x} = \mathbf{d}$.

If $\#$ equations $<$ $\#$ unknowns, then

- $A\mathbf{x} = \mathbf{0}$ has infinitely many solutions.
- for any given $\mathbf{d}$, $A\mathbf{x} = \mathbf{d}$ has 0 or infinitely many solutions.
- if rank$(A) = \#$ equations, $A\mathbf{x} = \mathbf{d}$ has infinitely many solutions for every $\mathbf{d}$.

Quiz

Consider the linear system of equations $A\mathbf{x} = \mathbf{d}$.

If $\#$ equations $>$ $\#$ unknowns, then

- $A\mathbf{x} = \mathbf{0}$ has one or infinitely many solutions.
- for any given $\mathbf{d}$, $A\mathbf{x} = \mathbf{d}$ has 0, 1, or infinitely many solutions.
- if rank$(A) = \#$ unknowns, $A\mathbf{x} = \mathbf{d}$ has 0 or 1 solution for every $\mathbf{d}$.

Quiz

Consider the linear system of equations $A\mathbf{x} = \mathbf{d}$.

If # equations = # unknowns, then

- $A\mathbf{x} = \mathbf{0}$ has one or infinitely many solutions.
- for any given $\mathbf{d}$, $A\mathbf{x} = \mathbf{d}$ has 0, 1, or infinitely many solutions.
- if rank(A) = # equations = #unknowns, $A\mathbf{x} = \mathbf{d}$ has exactly one solution for every $\mathbf{d}$.

Given A is a square matrix. Then

$$\begin{aligned} A\mathbf{x} &= \mathbf{d} \\ \Rightarrow \quad A^{-1}A\mathbf{x} &= A^{-1}\mathbf{d} \\ \Rightarrow \quad \mathbf{x} &= A^{-1}\mathbf{d} \end{aligned}$$

Q: When does a system of linear equations $A\mathbf{x} = \mathbf{d}$ have a unique solution **?**

A: A^{-1} exists (i.e., A is nonsingular).

Q: Show that $A\mathbf{x} = \mathbf{d}$ cannot have exactly two different solutions.

- The quantity $a_{11}a_{22} - a_{12}a_{21}$ is called the **determinant** of the 2 × 2 **square** matrix $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ and is composed of all the elements of A. It is denoted by $|A|$ or $\det(A)$.

ex: $\begin{vmatrix} 1 & 2 \\ 3 & -1 \end{vmatrix} = 1(-1) - 2(3) = -7$

- $$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}a_{22}a_{33} + a_{21}a_{32}a_{13} + a_{31}a_{23}a_{12} - a_{13}a_{22}a_{31} - a_{12}a_{21}a_{33} - a_{11}a_{32}a_{23}$$

- Determinants of order higher than 3 must be evaluated by **Laplace expansion**.

- Consider an $n \times n$ matrix, A, with typical element a_{ij}. The **minor** associated with each element is denoted M_{ij} and is the determinant of the $(n-1) \times (n-1)$ matrix formed by deleting the ith row and jth column of the matrix A.

ex: $$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}_{n \times n}$$

$$\Rightarrow M_{11} = \begin{vmatrix} a_{22} & a_{23} & \cdots & a_{2n} \\ a_{32} & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n2} & a_{n3} & \cdots & a_{nn} \end{vmatrix}_{(n-1) \times (n-1)}$$

- The **cofactor** of element a_{ij} is the minor of that element multiplied by $(-1)^{i+j}$, and is denoted C_{ij}:

$$C_{ij} = (-1)^{i+j} M_{ij}, \qquad i, j = 1, 2, \ldots, n$$

$$\begin{vmatrix} + & - & + & \cdots & (-1)^{1+n} \\ - & + & - & \cdots & (-1)^{2+n} \\ + & - & + & \cdots & (-1)^{3+n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ (-1)^{n+1} & (-1)^{n+2} & (-1)^{n+3} & \cdots & + \end{vmatrix}$$

$$A = \begin{bmatrix} 1 & 3 & 0 & 2 \\ -2 & -5 & 7 & 4 \\ 3 & 5 & 2 & 1 \\ -1 & 0 & -9 & -5 \end{bmatrix}$$

$$\Rightarrow M_{22} = \begin{vmatrix} 1 & 0 & 2 \\ 3 & 2 & 1 \\ -1 & -9 & -5 \end{vmatrix}, \quad C_{22} = M_{22}$$

$$A = \begin{bmatrix} 1 & 3 & 0 & 2 \\ -2 & -5 & 7 & 4 \\ 3 & 5 & 2 & 1 \\ -1 & 0 & -9 & -5 \end{bmatrix}$$

$$\Rightarrow M_{14} = \begin{vmatrix} -2 & -5 & 7 \\ 3 & 5 & 2 \\ -1 & 0 & -9 \end{vmatrix}, \quad C_{14} = -M_{14}$$

- The determinant of an $n \times n$ matrix A may be found by adding along any row or column the product of each element a_{ij} and its associated cofactor, that is,

$$|A| = \sum_{j=1}^{n} a_{ij} C_{ij} = \sum_{i=1}^{n} a_{ij} C_{ij}$$

by ith row by jth column

ex: $$\begin{vmatrix} 3 & 0 & 4 \\ 2 & 3 & 2 \\ 0 & 5 & -1 \end{vmatrix} = 3 \begin{vmatrix} 3 & 2 \\ 5 & -1 \end{vmatrix} - 0 \begin{vmatrix} 2 & 2 \\ 0 & -1 \end{vmatrix} + 4 \begin{vmatrix} 2 & 3 \\ 0 & 5 \end{vmatrix}$$

- Properties of Determinant

1. The interchange of rows and columns does not change the value of a determinant. $\Rightarrow \quad |A| = |A^T|$
2. The interchange of any two rows (columns) will alter the sign of the determinant.
3. The multiplication of any one row (column) by a scalar λ will change the value of the determinant λ-fold.
4. The addition (subtraction) of a multiple of any row (column) to (from) another row (column) will leave the value of the determinant unchanged.

5. The expansion of a determinant by **alien cofactors** (the cofactors of a "wrong" row or column) always yields zero.

$$\Rightarrow \sum_{j=1}^{n} a_{ij} C_{kj} = |A^*|$$

$$= |A|\text{'s } k\text{th row replaced by its } i\text{th row}$$

$\Rightarrow$ the kth row and the ith row in $|A^*|$ are identical

$$\Rightarrow |A^*| = 0$$

- An $n \times n$ matrix, A, has an associated **cofactor matrix** that is also $n \times n$ and is formed by replacing each a_{ij} with its associated cofactor.

$$\begin{bmatrix} C_{11} & C_{12} & \cdots & C_{1n} \\ C_{21} & C_{22} & \cdots & C_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ C_{n1} & C_{n2} & \cdots & C_{nn} \end{bmatrix}$$

- The **adjoint matrix** of an $n \times n$ matrix A, denoted adj(A), is the transpose of the cofactor matrix of A.
- The **inverse** of an $n \times n$ matrix A is the adjoint matrix of A divided by the determinant of A:
  $A^{-1} = \frac{1}{|A|}$ adj(A)

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \Rightarrow \text{adj}(A) = \begin{bmatrix} C_{11} & C_{21} & \cdots & C_{n1} \\ C_{12} & C_{22} & \cdots & C_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ C_{1n} & C_{2n} & \cdots & C_{nn} \end{bmatrix}$$

$$\Rightarrow A\ \text{adj}(A) = \begin{bmatrix} \sum_{j=1}^{n} a_{1j}C_{1j} & \sum_{j=1}^{n} a_{1j}C_{2j} & \cdots & \sum_{j=1}^{n} a_{1j}C_{nj} \\ \sum_{j=1}^{n} a_{2j}C_{1j} & \sum_{j=1}^{n} a_{2j}C_{2j} & \cdots & \sum_{j=1}^{n} a_{2j}C_{nj} \\ \vdots & \vdots & \ddots & \vdots \\ \sum_{j=1}^{n} a_{nj}C_{1j} & \sum_{j=1}^{n} a_{nj}C_{2j} & \cdots & \sum_{j=1}^{n} a_{nj}C_{nj} \end{bmatrix}$$

$$= \begin{bmatrix} |A| & & 0 \\ & \ddots & \\ 0 & & |A| \end{bmatrix} = |A|\ I_n$$

ex: $B = \begin{bmatrix} 1 & 0 \\ 9 & 2 \end{bmatrix} \qquad C = \begin{bmatrix} 4 & -2 & 1 \\ 7 & 3 & 3 \\ 2 & 0 & 1 \end{bmatrix}$

$$\Rightarrow \; B^{-1} = \frac{1}{|B|}\,\text{adj}(B) = \frac{1}{2}\begin{bmatrix} 2 & -9 \\ 0 & 1 \end{bmatrix}^T = \begin{bmatrix} 1 & 0 \\ -4.5 & 0.5 \end{bmatrix}$$

$$\Rightarrow \; C^{-1} = \frac{1}{|C|}\,\text{adj}(C)$$

$$= \frac{1}{8}\begin{bmatrix} 3 & -1 & -6 \\ 2 & 2 & -4 \\ -9 & -5 & 26 \end{bmatrix}^T = \frac{1}{8}\begin{bmatrix} 3 & 2 & -9 \\ -1 & 2 & -5 \\ -6 & -4 & 26 \end{bmatrix}$$

- $|A| \neq 0 \;\Leftrightarrow\; A^{-1}$ exists $\;\Leftrightarrow\; A$ is nonsingular
  $\Leftrightarrow\; A\mathbf{x} = \mathbf{d}$ has a unique solution.

- **Cramer's Rule**

$$A\mathbf{x} = \mathbf{d}$$
$$\Rightarrow \quad \mathbf{x} = A^{-1}\mathbf{d} = \frac{1}{|A|}\text{adj}(A)\mathbf{d}$$

$$\Rightarrow \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \frac{1}{|A|}\begin{bmatrix} C_{11} & \cdots & C_{n1} \\ \vdots & \ddots & \vdots \\ C_{1n} & \cdots & C_{nn} \end{bmatrix}\begin{bmatrix} d_1 \\ \vdots \\ d_n \end{bmatrix} = \frac{1}{|A|}\begin{bmatrix} \sum_{i=1}^{n} d_i C_{i1} \\ \vdots \\ \sum_{i=1}^{n} d_i C_{in} \end{bmatrix}$$

Note that $\sum_{i=1}^{n} d_i C_{ij}$ is nothing but the evaluation of the determinant derived from A by replacing its jth column by $\mathbf{d}$.

ex: $$\begin{bmatrix} 4 & -1 & 2 \\ 1 & 2 & -2 \\ -1 & 1 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 13 \\ 0 \\ 5 \end{bmatrix}$$

$$x = \frac{1}{\Delta} \begin{vmatrix} 13 & -1 & 2 \\ 0 & 2 & -2 \\ 5 & 1 & 1 \end{vmatrix}, \quad y = \frac{1}{\Delta} \begin{vmatrix} 4 & 13 & 2 \\ 1 & 0 & -2 \\ -1 & 5 & 1 \end{vmatrix},$$

$$z = \frac{1}{\Delta} \begin{vmatrix} 4 & -1 & 13 \\ 1 & 2 & 0 \\ -1 & 1 & 5 \end{vmatrix}, \quad \text{where } \Delta = \begin{vmatrix} 4 & -1 & 2 \\ 1 & 2 & -2 \\ -1 & 1 & -1 \end{vmatrix}$$

Determinant $\|A\|$ \ Vector $\mathbf{d}$		$\mathbf{d} \neq \mathbf{0}$ (nonhomogeneous system)	$\mathbf{d} = \mathbf{0}$ (homogeneous system)
$\|A\| \neq 0$ (A is nonsingular)		a unique, nontrivial solution $\mathbf{x} \neq \mathbf{0}$	a unique, trivial solution $\mathbf{x} = \mathbf{0}$
$\|A\| = 0$ (A is singular)	dependent	infinite number of solutions	infinite number of solutions
	inconsistent	no solution	not applicable

- A **triangular matrix** is composed of a nonzero element in the positions above (below) the main diagonal and zero in the positions below (above).

- The determinant of a triangular matrix equals the product of the diagonal elements.

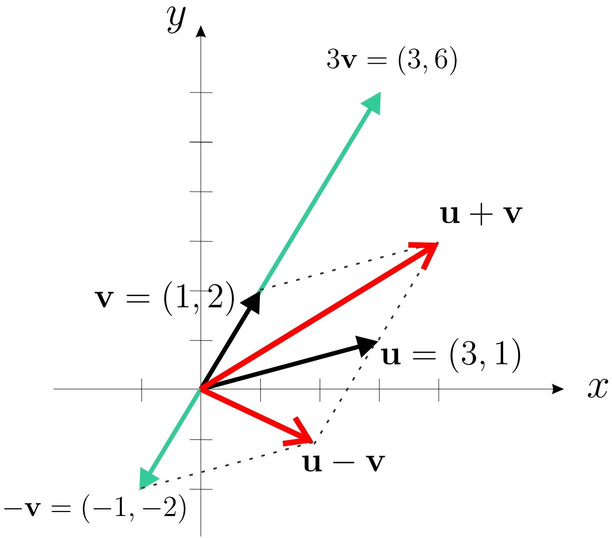
ex: $A = \begin{bmatrix} 1 & 0 \\ 2 & 2 \end{bmatrix} \Rightarrow |A| = \begin{vmatrix} 1 & 0 \\ 2 & 2 \end{vmatrix} = 2$

$$B = \begin{bmatrix} 4 & 0 & 0 \\ 7 & 3 & 0 \\ 2 & 1 & 5 \end{bmatrix} \Rightarrow |B| = \begin{vmatrix} 4 & 0 & 0 \\ 7 & 3 & 0 \\ 2 & 1 & 5 \end{vmatrix} = 60$$

- Let $\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} \in \mathbb{R}^n$ so that $\mathbf{v}^T = \begin{bmatrix} v_1 & v_2 & \cdots & v_n \end{bmatrix}$

- The **length** of an n-dimensional vector $\mathbf{v}$ is

$$\|\mathbf{v}\| = \sqrt{\mathbf{v}^T\mathbf{v}} = \sqrt{v_1^2 + v_2^2 + \cdots + v_n^2}$$



y
$3\mathbf{v} = (3, 6)$
$\mathbf{u} + \mathbf{v}$
$\mathbf{v} = (1, 2)$
$\mathbf{u} = (3, 1)$
x
$\mathbf{u} - \mathbf{v}$
$-\mathbf{v} = (-1, -2)$

- Two vectors in $\mathbb{R}^2$, $\mathbf{u}$ and $\mathbf{v}$, are **linearly independent** if

$$\lambda_1\mathbf{u} + \lambda_2\mathbf{v} = \mathbf{0}$$

holds only when the scalars λ_1 and λ_2 are both zero. Here $\mathbf{0}$ is the null vector.

- Otherwise, if there exist λ_1 and λ_2 are neither zero, then $\mathbf{u}$ and $\mathbf{v}$ would point in the same direction and be **linearly dependent**. That is,

$$\mathbf{u} = -\frac{\lambda_2}{\lambda_1}\mathbf{v}$$

- Any vector in $\mathbb{R}^2$ can be expressed as a **linear combination** of two linearly independent vectors in $\mathbb{R}^2$.

proof:

Given two linearly independent vectors, $\mathbf{v}$ and $\mathbf{w}$ in $\mathbb{R}^2$. For any vector $\mathbf{u}$, we write $\mathbf{u} = \lambda_1 \mathbf{v} + \lambda_2 \mathbf{w}$ and if $\lambda = \begin{bmatrix} \lambda_1 & \lambda_2 \end{bmatrix}^T$ has a solution, then the proof is done.

$$\lambda_1 \mathbf{v} + \lambda_2 \mathbf{w} = \begin{bmatrix} v_1 & w_1 \\ v_2 & w_2 \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix} = \mathbf{u}$$

Since $\mathbf{v}$ and $\mathbf{w}$ are linearly independent, $\begin{vmatrix} v_1 & w_1 \\ v_2 & w_2 \end{vmatrix} \neq 0$ which means λ has a solution.

- Let $\mathcal{V} = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ be a set of vectors in $\mathbb{R}^m$, then the vectors in $\mathcal{V}$ are **linearly dependent** iff

(i) some one of them can be expressed as a linear combination of the remaining vectors, or

(ii) there exists a set of scalars, $(\lambda_1, \lambda_2, \ldots, \lambda_n)$ (which are not all zero), such that

$$\sum_{i=1}^{n} \lambda_i \mathbf{v}_i = \lambda_1 \mathbf{v}_1 + \lambda_2 \mathbf{v}_2 + \cdots + \lambda_n \mathbf{v}_n = \mathbf{0}$$

- If $\sum_{i=1}^{n} \lambda_i \mathbf{v}_i = 0$ only holds when $\lambda_i = 0, \ \forall i$, then these vectors are **linearly independent**.

- If $\mathbf{v}$ and $\mathbf{w}$ are vectors in $\mathbb{R}^n$, then $\mathbf{v} + \mathbf{w}$ is a vector in $\mathbb{R}^n$ and so is $\lambda \mathbf{v}$. We say that $\mathbb{R}^n$ is a **vector space** for which addition and scalar multiplication can be defined and which is closed under these operations.

- Once we have found n linearly independent vectors in the n-space, all the other vectors in the space can be expressed as a linear combination of these n vectors.

- A **basis** is a set of linearly independent vectors that generates all vectors in the space.

ex: $\mathbf{e_1} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \mathbf{e_2} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \implies \mathbb{R}^2$

$$\mathbf{e_1} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \mathbf{e_2} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \mathbf{e_3} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \implies \mathbb{R}^3$$

Univariate Calculus and Optimization

refer to textbook

- A **sequence** of real numbers is an assignment of a real number to each natural number, usually written as $\{x_1, x_2, x_3, \ldots, x_n, \ldots\}$ or $\{x_n\}_{n=1}^{\infty}$.

ex: $\{1, 2, 3, 4, \ldots\}$ (F1.)

ex: $\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots\}$ (F2.)

ex: $\{1, \frac{1}{2}, 4, \frac{1}{8}, 16, \ldots\}$ (F3.)

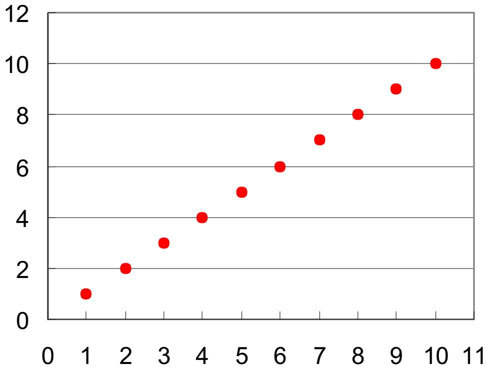
ex: $\{0, -\frac{1}{2}, \frac{2}{3}, -\frac{3}{4}, \frac{4}{5}, \ldots\}$ (F4.)

ex: $\{-1, 1, -1, 1, -1, \ldots\}$ (F5.)

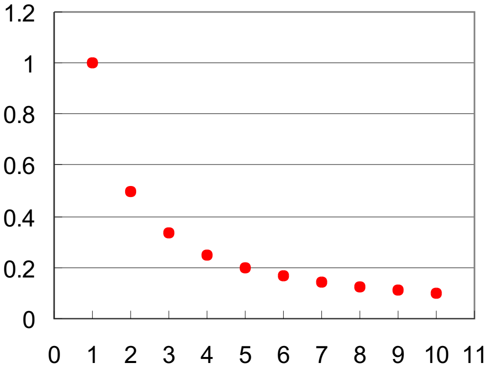
ex: $\{\frac{2}{1}, \frac{3}{2}, \frac{4}{3}, \frac{5}{4}, \ldots\}$ (F6.)

ex: $\{3.1, 3.14, 3.141, 3.1415, \ldots\}$ (F7.)

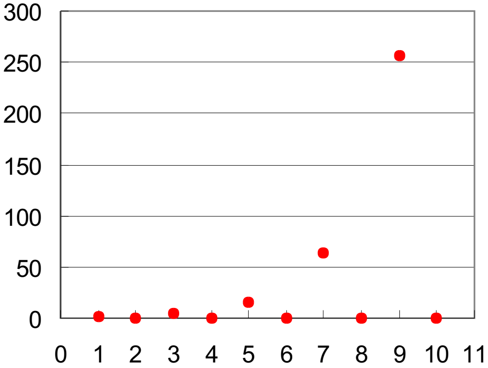
- $\{1, 2, 3, 4, \ldots\}$



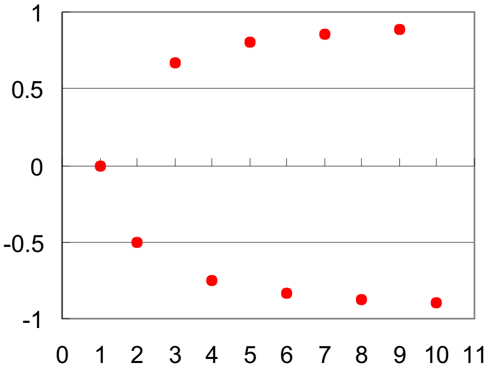
- $\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots\}$



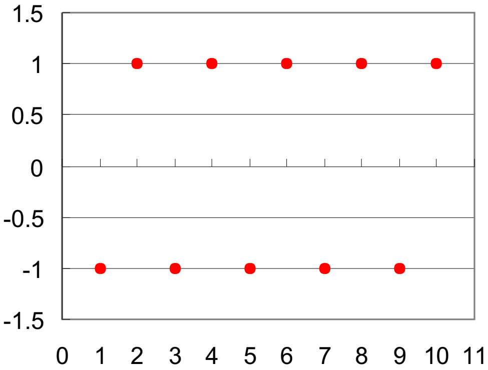
- $\{1, \frac{1}{2}, 4, \frac{1}{8}, 16, \ldots\}$



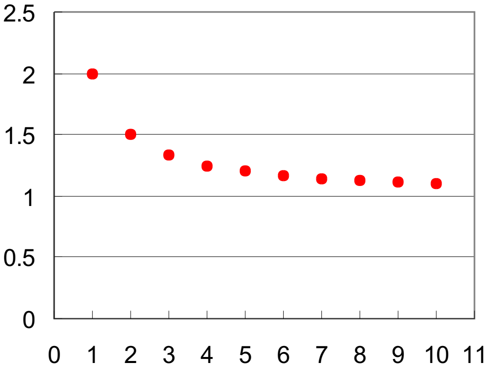
- $\{0, -\frac{1}{2}, \frac{2}{3}, -\frac{3}{4}, \frac{4}{5}, \ldots\}$



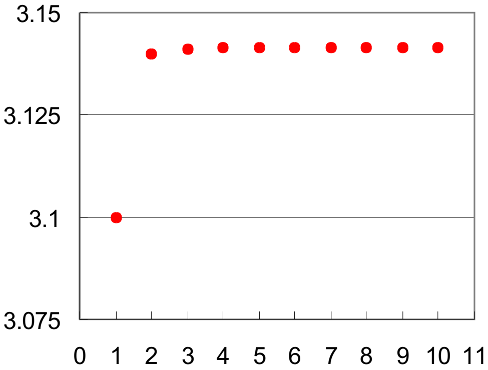
- $\{-1, 1, -1, 1, -1, \ldots\}$



- $\left\{\frac{2}{1}, \frac{3}{2}, \frac{4}{3}, \frac{5}{4}, \ldots\right\}$



- $\{3.1, 3.14, 3.141, 3.1415, \ldots\}$



There are basically **3** kinds of sequences:

- sequences in which the entries get closer and closer and stay close to some limiting value
- sequences in which the entries increase (or decrease) without bound
- sequences in which the entries jump back and forth on the number line

- Let $\{x_n\}_{n=1}^{\infty}$ be a sequence of real numbers and let r be a real number. We say that r is the **limit** of this sequence if for any (small) positive number ϵ, there is a positive integer N such that for all $n \geq N$, x_n is in the ϵ-interval about r, i.e.,

$$|x_n - r| < \epsilon,$$

then we say that the sequence **converges to** r and write

$$\lim x_n = r \quad \text{or} \quad \lim_{n\to\infty} x_n = r \quad \text{or} \quad x_n \to r.$$

Note

1. The elements of the converging sequence need not be distinct from each other or distinct from the limit.
2. The convergence need not be all from one side.
3. The convergence need not be **monotonic**: each element need not be closer to the limit than all previous elements.

accumulation point (or cluster point)

If for any positive ϵ there are infinitely many elements of the sequence in the interval $I_\epsilon(r)$, then r is a cluster point of the sequence.

- A sequence can have at most one limit.

  Proof: Suppose that a sequence $\{x_n\}_{n=1}^{\infty}$ has two limits: r_1 and r_2. Take ϵ to be some number less than $\frac{1}{2}|r_1 - r_2|$, say $\epsilon = \frac{1}{4}|r_1 - r_2|$, so that $I_\epsilon(r_1)$ and $I_\epsilon(r_2)$ are disjoint intervals.

  Since $x_n \to r_1$, there is an N_1 such that for $n \geq N_1$ all the x_n are in $I_\epsilon(r_1)$. Similarly, there is an N_2 such that for $n \geq N_2$ all the x_n are in $I_\epsilon(r_2)$. Hence, for all $n \geq \max\{N_1, N_2\}$, x_n are in both $I_\epsilon(r_1)$ and $I_\epsilon(r_2)$.

  But no point can be in both two disjoint intervals $\Rightarrow$ **Contradiction!**

- When we say $x \to a$, the variable x can approach the number a either from values less than a (written $x \to a^-$), or from values greater than a (written $x \to a^+$).
- If, as $x \to a$ from the left side, the function $f(x)$ approaches a finite number L_1, written

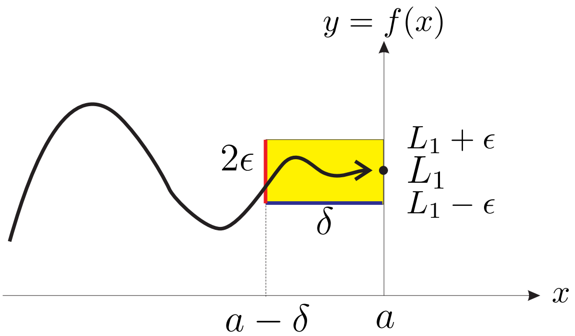
$$\lim_{x \to a^-} f(x) = L_1,$$

then we call L_1 the **left-hand limit of $f(x)$ at $x = a$**.

- If, as $x \to a$ from the right side, the function $f(x)$ approaches a finite number L_2, written

$$\lim_{x \to a^+} f(x) = L_2,$$

then we call L_2 the **right-hand limit of $f(x)$ at $x = a$**.



- If for any $\epsilon > 0$, however small, there exists some $\delta > 0$, such that $|f(x) - L_1| < \epsilon$, $\forall\ x$ satisfying $a - \delta < x < a$, then the left-hand limit **exists** and is equal to L_1.
- If for any $\epsilon > 0$, however small, there exists some $\delta > 0$, such that $|f(x) - L_2| < \epsilon$, $\forall\ x$ satisfying $a < x < a + \delta$, then the right-hand limit **exists** and is equal to L_2

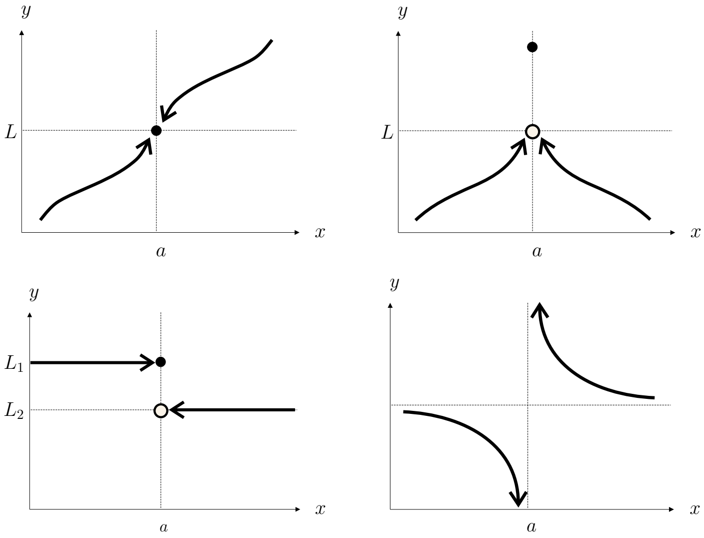
- Suppose that a function $y = f(x)$ is defined on some open interval including the point a. We say that **the limit of $f(x)$ at $x = a$**, that is, $\lim_{x \to a} f(x)$, **exists** if

  (i) $L_1 = \lim_{x \to a^-} f(x)$ and $L_2 = \lim_{x \to a^+} f(x)$ exist

  and

  (ii) $L_1 = L_2 = L$.

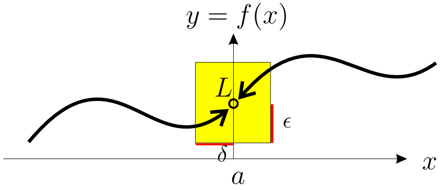
- Note that $\lim_{x \to a} f(x)$ (the limit of $f(x)$ at $x = a$) is distinct from $f(a)$ (the function value of $f(x)$ at $x = a$).


y
L
x
a
y
L
x
a
y
L_1
L_2
x
a
y
x
a

The Formal Definition of Limit

- As $x \to a$, the **limit** of $f(x)$ is the finite number L if, given any positive ϵ (however small), there can be found a positive number δ such that

$$|f(x) - L| < \epsilon \quad \text{for} \quad 0 < |x - a| < \delta$$



Limit Theorems

- If $\lim_{x \to a} f(x) = f_0$ and $\lim_{x \to a} g(x) = g_0$, then

  (1) $\lim_{x \to a} [\ f(x) \pm g(x)\ ] = f_0 \pm g_0$

  (2) $\lim_{x \to a} f(x) g(x) = f_0\ g_0$ (3) $\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{f_0}{g_0}$, $(g_0 \neq 0)$

  ex: $\lim_{x \to a} x = a$ **ex:** $\lim_{x \to a} k = k$

  ex: $\lim_{x \to a} \gamma x + \delta = \lim_{x \to a} \gamma \lim_{x \to a} x + \lim_{x \to a} \delta = \gamma a + \delta$

  ex: $\lim_{x \to a} x^n = (\lim_{x \to a} x)^n = a^n$

- A function $f(x)$, which is defined on an open interval including the point $x = a$, is **continuous at a** if

  (i) $\lim_{x \to a} f(x)$ exists and (ii) $\lim_{x \to a} f(x) = f(a)$.

- A function $f(x)$, which is defined on an open interval including the point $x = a$, is continuous at that point if, given any positive ϵ (however small), there can be found a positive number δ such that $|f(x) - f(a)| < \epsilon$, whenever $|x - a| < \delta$.

- A function that is **not** continuous is said to be **discontinuous**.

- Suppose that $f(x)$ and $g(x)$ are continuous functions and that $c \neq 0$ is a constant. The following are also continuous:

(i) $cf(x)$

(ii) $f(x) + c$

(iii) $f(x) \pm g(x)$

(iv) $f(x)g(x)$

(v) $f(x)/g(x)$ for $g(x) \neq 0$

(vi) $f^{-1}(\cdot)$, if it exists

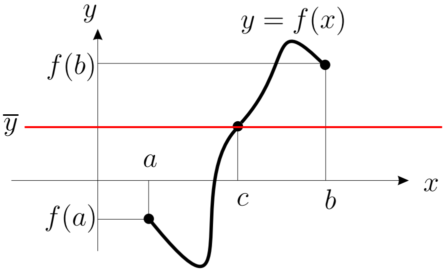
- Let $f(x)$ be defined on the closed interval $[a, b], x \in \mathbb{R}$ and $a < b$. We say that

  (i) $f(x)$ is **continuous from the right** at the point $x = a$ if $\lim_{x \to a^+} f(x)$ exists, $f(a)$ exists, and $\lim_{x \to a^+} f(x) = f(a)$.

  (ii) $f(x)$ is **continuous from the left** at the point $x = b$ if $\lim_{x \to b^-} f(x)$ exists, $f(b)$ exists, and $\lim_{x \to b^-} f(x) = f(b)$.

  (iii) $f(x)$ is continuous on the closed interval $[a, b]$ if it is (1) continuous at every point x strictly within the interval (i.e., $a < x < b$), (2) continuous from the right at $x = a$ and (3) continuous from the left at $x = b$.

- **(Intermediate-value theorem)**
  Suppose that $f(x)$ is a **continuous** function on the closed interval $[a, b]$ and that $f(a) \neq f(b)$. Then, for any number $\overline{y}$ between $f(a)$ and $f(b)$, there is some value of x, say $x = c$, between a and b such that $\overline{y} = f(c)$.



ex: If the demand and supply functions are continuous and the following two conditions are satisfied:

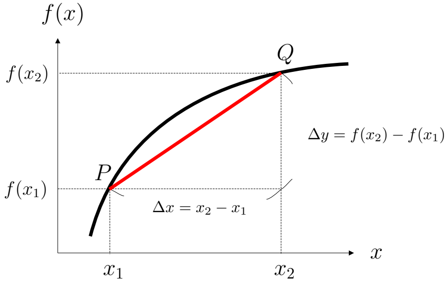
(i) at zero price, $D(0) > S(0)$,

(ii) there exists some price, $\hat{p} > 0$, at which $S(\hat{p}) > D(\hat{p})$,

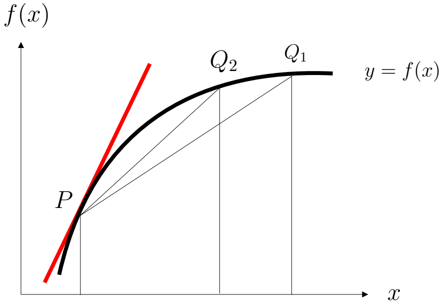
then there exists a positive equilibrium price in the market.

Hint: Let $Z(p) = D(p) - S(p)$

- Given two points $P = (x_1, f(x_1))$ and $Q = (x_2, f(x_2))$ on the graph of a function $y = f(x)$, we define the **secant line** as the straight line joining these two points and its slope is $m_{PQ} = \frac{f(x_2)-f(x_1)}{x_2-x_1} = \frac{\Delta y}{\Delta x}$



- If the function $y = f(x)$ is defined on some open interval including the point $P = (x_1, f(x_1))$ and $\lim_{\Delta x \to 0} m_{PQ}$ exists, then the line passing through the point P with slope equal to $\lim_{\Delta x \to 0} m_{PQ}$ is the **tangent line** of the function $y = f(x)$ at P.

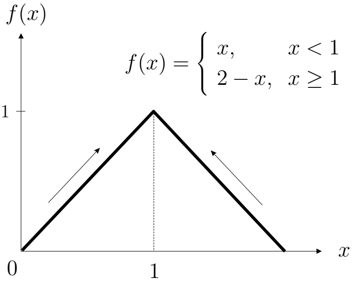


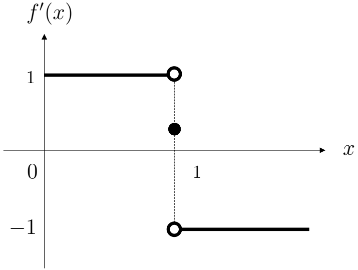
- The **derivative** of a function $y = f(x)$ at the point $P = (x_1, f(x_1))$ is the slope of the tangent line at that point.

$$f'(x_1) = \lim_{\Delta x \to 0} m_{PQ} = \lim_{x_2 \to x_1} \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

where $\Delta x = x_2 - x_1$. We can also write this as

$$f'(x_1) = \lim_{\Delta x \to 0} m_{PQ} = \lim_{\Delta x \to 0} \frac{f(x_1 + \Delta x) - f(x_1)}{\Delta x}$$


$f(x)$
$f(x) = \begin{cases} x, & x < 1 \\ 2 - x, & x \geq 1 \end{cases}$
1
0
1
x


$f'(x)$
1
x
0
1
-1

- If $f'(x)$ exists (i.e., the function $f(x)$ is **differentiable**) at the point $x = a$, then the function $f(x)$ must also be continuous at this point.

Proof:

$$\lim_{x \to a}[\ f(x) - f(a)\ ] = \lim_{x \to a} f(x) - f(a)$$

$$\lim_{x \to a}[\ \frac{f(x) - f(a)}{x - a}(x - a)\ ] = \lim_{x \to a}[\ \frac{f(x) - f(a)}{x - a}\ ] \lim_{x \to a}(x - a)$$

$$= f'(a)(\lim_{x \to a} x - a) = 0$$

- The smoothness of a primitive function, $f(x)$, can be linked to the continuity of its derivative function, $f'(x)$. That is, if a certain function is smooth everywhere on the domain, it is referred to as a continuously differentiable function.

- A function $f(x)$ defined on the domain $x \in [a, b]$ is differentiable on $[a, b]$ if

  (1) the right-hand derivative for $f(x)$ exists at $x = a$,
  (2) the left-hand derivative exists at $x = b$,
  (3) $f(x)$ is differentiable at every point in the open set (a, b).

Rules of Differentiation

- $f(x) = k$ (a constant) $\quad \Rightarrow \quad f'(x) = 0$
- $f(x) = x^n \quad \Rightarrow \quad f'(x) = nx^{n-1}$
- $\frac{d}{dx}[\ f(x) \pm g(x)\ ] = f'(x) \pm g'(x)$

ex: $f(x) = 4x^4 - x^3 + 17x^2 + 3x - 1$

$f'(x) = 16x^3 - 3x^2 + 34x + 3$

$f''(x) = 48x^2 - 6x + 34$

$f'''(x) = 96x - 6 \qquad f^{(4)}(x) = 96 \qquad f^{(5)}(x) = 0$

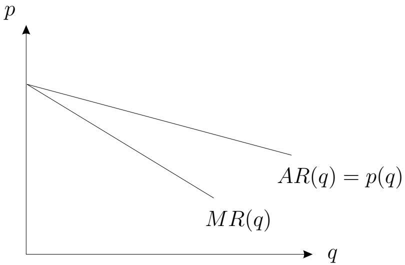
- $\frac{d}{dx}[\ f(x)g(x)\ ] = f'(x)g(x) + f(x)g'(x)$

ex: $\frac{d}{dx}[(2x+3)(3x^2)] = (2)(3x^2) + (2x+3)(6x) = 18x^2 + 18x$

- $\frac{d}{dx}[\ f(x)g(x)h(x)\ ] =$
$$f'(x)g(x)h(x) + f(x)g'(x)h(x) + f(x)g(x)h'(x)$$

- $\frac{d}{dx}[\ \frac{f(x)}{g(x)}\ ] = \frac{f'(x)g(x) - f(x)g'(x)}{[\ g(x)\ ]^2}$

ex: $\frac{d}{dx}\left(\frac{2x-3}{x+1}\right) = \frac{(2)(x+1) - (2x-3)(1)}{(x+1)^2} = \frac{5}{(x+1)^2}$



ex:

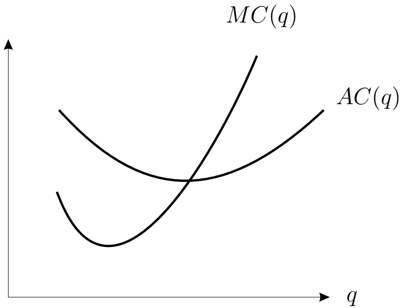
$$AR(q) = P(q)$$

$$\Rightarrow TR(q) = AR(q)q = P(q)q$$

$$\Rightarrow MR(q) = \frac{d}{dq}TR(q)$$

$$= P'(q)q + P(q)$$

$$\Rightarrow MR(q) - AR(q) = P'(q)q < 0$$



ex:

$$AC(q) = \frac{TC(q)}{q}$$

$$\Rightarrow \frac{d}{dq}AC(q) = \frac{d}{dq}\left[\frac{TC(q)}{q} \right]$$

$$= \frac{\left[\frac{d}{dq}TC(q) \right]q - TC(q)}{q^2}$$

$$= \frac{1}{q}\left[MC(q) - AC(q) \right]$$

The Chain Rule

- If $y = f(u)$ and $u = g(x)$ so that $y = f(g(x)) = h(x)$, then

$$h'(x) = f'(u)g'(x) \quad \text{or} \quad \frac{dy}{dx} = \left(\frac{dy}{du}\right)\left(\frac{du}{dx}\right)$$

ex:

$TR = TR(q)$ and $q = q(L)$ so that $TR = f(L)$

$$\Rightarrow MRP(L) = \frac{d}{dL}f(L) = \left(\frac{dTR(q)}{dq}\right)\left(\frac{dq(L)}{dL}\right)$$

$$= MR(q)MP(L)$$

The Derivative of the Inverse of a Function

- If $y = f(x)$ has the inverse function $x = f^{-1}(y) = g(y)$, then

$$\frac{dx}{dy} = \frac{1}{dy/dx} \quad \text{or} \quad g'(y) = \frac{1}{f'(x)}$$

ex:

$TC(L) = wL + C_0$ and $q = q(L)$ (or $L = L(q)$)

$$\Rightarrow \; TC(q) = wL(q) + C_0$$

$$\Rightarrow \; MC(q) = \frac{d}{dq}TC(q) = w\frac{dL(q)}{dq} = \frac{w}{dq(L)/dL} = \frac{w}{MP(L)}$$

- For a function $y = f(x)$, which is assumed to be nth-order continuously differentiable,

  (i) the first derivative function (the slope of f):

$$f'(x) = \frac{dy}{dx}$$

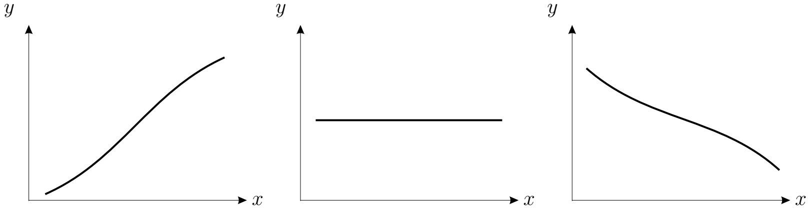
  (ii) the second derivative function (the rate of change of the slope of f):

$$f''(x) = \frac{d}{dx}[\ f'(x)\ ] = \frac{d}{dx}[\ \frac{dy}{dx}\ ] = \frac{d^2y}{dx^2}$$

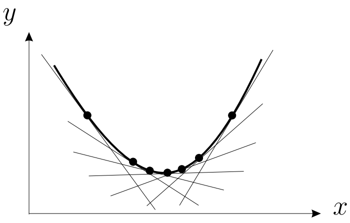
  (iii) the third derivative function:

$$f'''(x) = \frac{d}{dx}[\ f''(x)\ ] = \frac{d^2}{dx^2}[\ f'(x)\ ] = \frac{d^3y}{dx^3}$$

- $f' > 0$: the **value** of f tends to **increase**
  $f' = 0$: the **value** of f tends to **stay constant**
  $f' < 0$: the **value** of f tends to **decrease**



- $f'' > 0$: the **slope** of the curve tends to **increase**
  $f'' < 0$: the **slope** of the curve tends to **decrease**



- **Objective function** $\Longrightarrow$ dependent variable

ex: Utility Maximization
Profit Maximization
Cost Minimization

- **Choice variable** $\Longrightarrow$ independent variable

ex: the quantities of goods
the quantities of products
the quantities of inputs

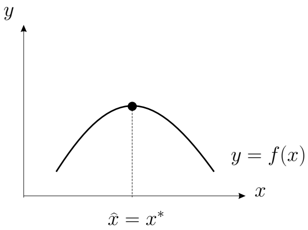
- At a **global (absolute) maximum** x^*,

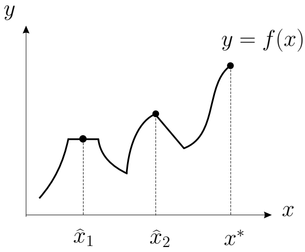
$$f(x^*) \geq f(x) \qquad \forall\ x$$

whereas at a **local (relative) maximum** $\hat{x}$,

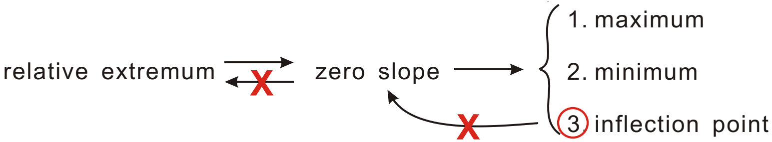
$$f(\hat{x}) \geq f(x), \qquad \forall\ x \in (\hat{x} - \epsilon, \hat{x} + \epsilon)$$

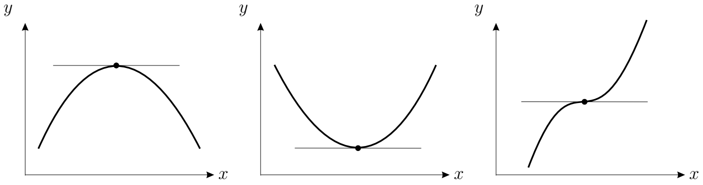
where ϵ (perhaps very small) is positive.





- If the **differentiable** function f takes an (local) extreme value (maximum or minimum) at a point x^*, then
  $f'(x^*) = 0$ [**first-order condition**].
- Note that the first-order condition, $f'(x^*) = 0$, is only **necessary** but **not sufficient** for x^* to yield an extremum value.





- If $f'(x^*) = 0$, then x^* : critical value

$$f(x^*) : \text{stationary value}$$

$$(x^*, f(x^*)) : \text{stationary point}$$

- A twice differentiable function $f(x)$ is **convex** (concave) if $f''(x) \geq 0$ ($f''(x) \leq 0$) at all points on its domain.
- A twice differentiable function $f(x)$ is **strictly convex** (strictly concave) if $f''(x) > 0$ ($f''(x) < 0$).
- However, $f''(x)$ might be zero at a stationary point for a strictly convex (strictly concave) function.

ex: $y = f(x) = x^4$ when considering $x = 0$.

- Hence, $f''(x^*) > (<)\ 0$ with $f'(x^*) = 0$ is **sufficient** but **not necessary** for $f(x^*)$ to be a relative minimum (maximum). It is **necessary** that $f''(x^*) \geq (\leq)\ 0$ with $f'(x^*) = 0$.

ex: Let the $R(Q)$ and $C(Q)$ functions be

$$R(Q) = 1200Q - 2Q^2$$
$$C(Q) = Q^3 - 61.25Q^2 + 1528.5Q + 2000$$

Then the profit function is

$$\pi(Q) = -Q^3 + 59.25Q^2 - 328.5Q - 2000$$

which has two critical values, $Q = 3$ and $Q = 36.5$, because

$$\frac{d\pi}{dQ} = -3Q^2 + 118.5Q - 328.5 = -3(Q-3)(Q-36.5).$$

But since the second derivative is

$$\frac{d^2\pi}{dQ^2} = -6Q + 118.5 \quad \begin{cases} > 0 & \text{when } Q = 3 \\ < 0 & \text{when } Q = 36.5 \end{cases}$$

the profit-maximizing output is $Q^* = 36.5$.

- **Maclaurin Series Expansion** of a Polynomial Function

$$
\begin{aligned}
f(x) &= a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots + a_n x^n & \Rightarrow \quad & f(0) = a_0 \\
f'(x) &= a_1 + 2a_2 x + 3a_3 x^2 + \cdots + n a_n x^{n-1} & \Rightarrow \quad & f'(0) = a_1 \\
f''(x) &= 2a_2 + (3)(2)a_3 x + \cdots + n(n-1)a_n x^{n-2} & \Rightarrow \quad & f''(0) = 2a_2 \\
& \vdots & & \\
f^{(n)}(x) &= n(n-1)(n-2)\cdots(3)(2)(1)a_n & \Rightarrow \quad & f^{(n)}(0) = n!\, a_n
\end{aligned}
$$

$$\Longrightarrow \quad f(x) = \frac{f(0)}{0!} + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \cdots + \frac{f^{(n)}(0)}{n!}x^n$$

- **Taylor Series Expansion** (around $x = x_0$)

  Let $x = x_0 + \delta \quad \Rightarrow \quad f(x) = f(x_0 + \delta) \equiv g(\delta)$

  Hence, $f'(x_0 + \delta) = g'(\delta)$ and $f^{(n)}(x_0 + \delta) = g^{(n)}(\delta)$

$$f(x) = g(\delta) = \frac{g(0)}{0!} + \frac{g'(0)}{1!}\delta + \frac{g''(0)}{2!}\delta^2 + \cdots + \frac{g^{(n)}(0)}{n!}\delta^n$$

$$= \frac{f(x_0)}{0!} + \frac{f'(x_0)}{1!}(x - x_0) + \frac{f''(x_0)}{2!}(x - x_0)^2 + \cdots + \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n$$

$$= \sum_{k=0}^{n} \left[\frac{f^{(k)}(x_0)}{k!}(x - x_0)^k \right]$$

- **Taylor's Theorem**

  Given an arbitrary function $f(x)$, if we know the values $f(x_0)$, $f'(x_0)$, $f''(x_0)$, $\cdots$, etc., then $f(x)$ can be expanded around x_0 as

$$f(x) = \left[\frac{f(x_0)}{0!} + \frac{f'(x_0)}{1!}(x - x_0) + \cdots + \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n\right] + R_{n+1}$$

$$= P_n + R_{n+1}$$

  where $R_{n+1} = \dfrac{f^{(n+1)}(p)}{(n+1)!}(x - x_0)^{n+1}$ and $p \in (x, x_0)$.

- If it happens that

  $R_{n+1} \to 0$ as $n \to \infty$ so that $P_n \to f(x)$ as $n \to \infty$

- A function $f(x)$ attains a relative **maximum** (minimum) value at x_0 if $f(x) - f(x_0)$ is **negative** (positive) for values of x in the immediate neighborhood of x_0.

- Because of the continuity of the nth derivative, $f^{(n)}(p)$ will have the same sign as $f^{(n)}(x_0)$ does since p is very close to x_0.

ex: $f'(x_0) \neq 0$

$$f(x) - f(x_0) = \frac{f'(p)}{1!}(x - x_0) = f'(p)(x - x_0)$$

$\Rightarrow$ $f(x_0)$ cannot be a relative extremum.

ex: $f'(x_0) = 0,\ f''(x_0) \neq 0$

$$f(x) - f(x_0) = \frac{f'(x_0)}{1!}(x - x_0) + \frac{f''(p)}{2!}(x - x_0)^2$$

$$= \frac{1}{2} f''(p)(x - x_0)^2$$

$\Rightarrow$ $f(x_0)$ is a relative maximum if $f''(x_0) < 0$ with $f'(x_0) = 0$.

ex: $f'(x_0) = f''(x_0) = 0,\ f'''(x_0) \neq 0$

$$f(x) - f(x_0) = \frac{f'(x_0)}{1!}(x - x_0) + \frac{f''(x_0)}{2!}(x - x_0)^2 + \frac{f'''(p)}{3!}(x - x_0)^3$$

$$= \frac{1}{6} f'''(p)(x - x_0)^3$$

$\Rightarrow$ $(x_0,\ f(x_0))$ is an inflection point.

- **Nth-Derivative Test**

  If $f'(x_0) = 0$ and the first nonzero derivative value at x_0 encountered in successive derivative is Nth, i.e., $f^{(N)}(x_0) \neq 0$, then the stationary value $f(x_0)$ will be

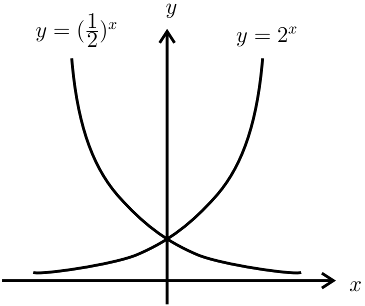
  1. a relative maximum if N is even and $f^{(N)}(x_0) < 0$.
  2. a relative minimum if N is even and $f^{(N)}(x_0) > 0$.
  3. an inflection point if N is odd.

  ex: $y = f(x) = x^3$

  ex: $y = f(x) = (x-2)^4 + 3$

- **Exponential Functions:**

$$y = f(x) = a^x,\ a > 0,\ a \neq 1.$$



Q: What kind of number a can, as a base of the exponential function $f(x) = a^x$,possess the property that $f(x) = f'(x)$ **?**

$$\Rightarrow f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{a^{x+h} - a^x}{h}$$

$$= a^x \lim_{h \to 0} \frac{a^h - 1}{h}$$

? $= a^x = f(x)$

$$\Rightarrow \lim_{h \to 0} \frac{a^h - 1}{h} = 1$$

$\Rightarrow$ Let $E(m) = (1 + \frac{1}{m})^m$, then

$$E(1) = 2,$$
$$E(2) = 2.25,$$
$$E(3) = 2.37037\cdots,$$
$$E(4) = 2.4414\cdots,$$
$$E(5) = 2.48832,$$
$$\vdots$$

$$\Rightarrow e \equiv \lim_{m\to\infty} E(m) = \lim_{m\to\infty}(1 + \frac{1}{m})^m \doteqdot 2.71828$$

$$\Rightarrow \frac{d}{dx}e^x = e^x$$

$$f(x) = e^x$$

$$\Rightarrow \; f(x) = f'(x) = f''(x) = \cdots = f^{(n)}(x) = e^x$$

$$\Rightarrow \; f(0) = f'(0) = f''(0) = \cdots = f^{(n)}(0) = 1$$

$$\Rightarrow \; e^x = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \cdots$$

$$= 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \cdots$$

$$\Rightarrow \; e = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \cdots \doteqdot 2.71828$$

- **Economic Interpretation**

(1) As the year-end value to which a principle of $\$1$ will grow if interest at the rate of 100% per annum is compounded continuously.

$$\Rightarrow V(1) = (1 + \frac{1}{1})^1,$$
$$V(2) = (1 + \frac{1}{2})^2,$$
$$V(3) = (1 + \frac{1}{3})^3,$$
$$\vdots$$
$$\Rightarrow \lim_{m\to\infty} V(m) = \lim_{m\to\infty} (1 + \frac{1}{m})^m = e$$

(2) As the t year-end value to which a principle of $\$A$ will grow if interest at the rate of r per annum is compounded continuously.

$$\Rightarrow V(1) = A(1+r)^t,$$

$$V(2) = A(1+\frac{r}{2})^{2t},$$

$$V(3) = A(1+\frac{r}{3})^{3t},$$

$$\vdots$$

$$\Rightarrow \lim_{m\to\infty} V(m) = \lim_{m\to\infty} A(1+\frac{r}{m})^{mt}$$

$$= \lim_{(m/r)\to\infty} A(1+\frac{1}{(m/r)})^{(m/r)rt}$$

$$= Ae^{rt}$$

(3) r as the instantaneous rate of growth of Ae^{rt}.

Let $V = Ae^{rt}$, then $\frac{dV}{dt} = Are^{rt} = rV$

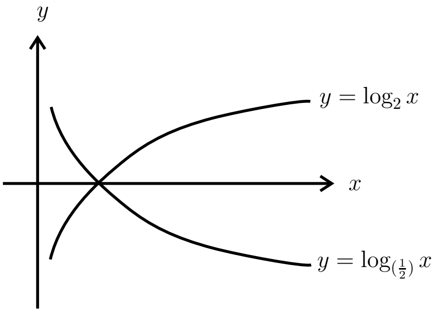
$$\Rightarrow \gamma_V = \frac{dV/dt}{V} = r.$$

(4) Discounting and the present value.

$$V = Ae^{rt} \quad \Rightarrow \quad A = Ve^{-rt}$$

- **Logarithms:**

$$y = f(x) = \log_a x, \ a > 0, \ a \neq 1, \ x > 0$$



- **Rules**

1. $\log_a(uv) = \log_a u + \log_a v$ **ex:** $\log_2 6 = \log_2 2 + \log_2 3$
2. $\log_a(\frac{u}{v}) = \log_a u - \log_a v$ **ex:** $\log_2 5 = \log_2 10 - \log_2 2$
3. $\log_a u^n = n \log_a u$ **ex:** $\log_{10} 0.001 = \log_{10} 10^{-3} = -3$
4. $\log_b u = (\log_b a)(\log_a u)$ **ex:** $(\log_4 3)(\log_3 64) = \log_4 4^3 = 3$
5. $\log_a u = (\log_u a)^{-1}$ **ex:** $\log_3 2 = \dfrac{1}{\log_2 3}$
6. $\log_{a^k} u^n = \frac{n}{k} \log_a u$ **ex:** $\log_4 8 = \log_{2^2} 2^3 = \frac{3}{2}$

- Define $\log_e x = \ln x$ as the **natural** logarithm and $\log_{10} x = \log x$ as the **common** logarithm.
- $\frac{d}{dx}\ln x = \frac{1}{x}$

proof:

Let $f(x) = \ln x$ and $m = \frac{x}{h}$

$$\Rightarrow f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\frac{\ln(\frac{x+h}{x})}{h}$$

$$= \lim_{m\to\infty}\frac{\ln(1+\frac{1}{m})}{(\frac{x}{m})} = \frac{1}{x}\lim_{m\to\infty}\ln(1+\frac{1}{m})^m$$

$$= \frac{1}{x}\ln(\lim_{m\to\infty}(1+\frac{1}{m})^m) = \frac{1}{x}$$

ex: $y = e^{3t}$

$$\Rightarrow \; y' = \frac{d}{dt}e^{3t} = (\frac{d}{d(3t)}e^{3t})(\frac{d(3t)}{dt}) = 3e^{3t}$$

ex: $y = \ln t^5$

$$\Rightarrow \; y' = (\frac{d}{d(t^5)}\ln t^5)(\frac{d}{dt}t^5) = \frac{1}{t^5}(5t^4) = \frac{5}{t}$$

ex: $y = t^3 \ln t^2$

$$\Rightarrow \; y' = (3t^2)(\ln t^2) + (t^3)(\frac{2}{t}) = 6t^2 \ln t + 2t^2$$

- $b = e^{\ln b}$ or $b = a^{\log_a b}$
- $\frac{d}{dx} b^x = b^x \ln b$
- $\frac{d}{dx} \log_b x = \frac{1}{x \ln b}$

proof 1:

$$\frac{d}{dx} b^x = \frac{d}{dx} (e^{\ln b})^x = \frac{d}{dx} e^{(\ln b)x} = (\ln b) e^{(\ln b)x} = b^x \ln b$$

proof 2:

$$\frac{d}{dx} \log_b x = \frac{d}{dx} (\frac{\ln x}{\ln b}) = \frac{1}{x \ln b}$$

ex: $y = 12^{1-t}$

$$\Rightarrow y' = (-1)12^{1-t}\ln 12$$

ex: $y = \dfrac{x^2}{(x+3)(2x+1)}$

$$\Rightarrow \ln y = \ln x^2 - \ln(x+3) - \ln(2x+1)$$

$$\Rightarrow (\frac{1}{y})y' = \frac{2}{x} - \frac{1}{x+3} - \frac{2}{2x+1}$$

$$\Rightarrow y' = \frac{x^2}{(x+3)(2x+1)}\left(\frac{2}{x} - \frac{1}{x+3} - \frac{2}{2x+1}\right)$$

ex: $y = 4^t \;\Rightarrow\; \ln y = \ln 4^t = t \ln 4$

$$\Rightarrow\; \frac{d}{dt}\ln y = \frac{1}{y}\left(\frac{dy}{dt}\right) \equiv \gamma_y = \ln 4$$

ex: $y = uv \;\Rightarrow\; \ln y = \ln u + \ln v \;\Rightarrow\; \gamma_y = \gamma_u + \gamma_v$

$y = \frac{u}{v} \;\Rightarrow\; \ln y = \ln u - \ln v \;\Rightarrow\; \gamma_y = \gamma_u - \gamma_v$

$y = u + v \;\Rightarrow\; \ln y = \ln(u + v)$

$$\Rightarrow\; \gamma_y = \frac{1}{u+v}\left(\frac{du}{dt} + \frac{dv}{dt}\right) = \frac{u}{u+v}\gamma_u + \frac{v}{u+v}\gamma_v$$

Multivariate Calculus and Optimization

refer to textbook

Ch.11 The Case of More than One Choice Variable

- Let $y = f(x_1, x_2, \cdots, x_n)$, where x_i are mutually independent. The **partial derivative** of y with respect to the variable x_i is

$$f_i \equiv \frac{\partial y}{\partial x_i}$$

$$= \lim_{\Delta x_i \to 0} \frac{f(x_1, \ldots, x_i + \Delta x_i, \ldots, x_n) - f(x_1, \ldots, x_i, \ldots, x_n)}{\Delta x_i}$$

ex: $f(x_1, x_2) = 3x_1^2 + x_1 x_2 + 4x_2^2$

$$\Rightarrow f_1(x_1, x_2) = 6x_1 + x_2 \quad \text{and} \quad f_2(x_1, x_2) = x_1 + 8x_2$$

ex: $f(x, y) = \dfrac{2x - 3y}{x + y}$

$$\Rightarrow f_x(x, y) = \frac{2(x + y) - (2x - 3y)}{(x + y)^2} = \frac{5y}{(x + y)^2}$$

$$f_y(x, y) = \frac{(-3)(x + y) - (2x - 3y)}{(x + y)^2} = \frac{-5x}{(x + y)^2}$$

ex: $Q^D = a - bP \quad (a, b > 0)$

$Q^S = -c + dP \quad (c, d > 0)$

$\Rightarrow \ P^* = \frac{a+c}{b+d}, \quad Q^* = \frac{ad-bc}{b+d}$

$\Rightarrow \ \frac{\partial P^*}{\partial a} = ?, \quad \frac{\partial P^*}{\partial b} = ?, \quad \frac{\partial P^*}{\partial c} = ?, \quad \frac{\partial P^*}{\partial d} = ?$

$\frac{\partial Q^*}{\partial a} = ?, \quad \frac{\partial Q^*}{\partial b} = ?, \quad \frac{\partial Q^*}{\partial c} = ?, \quad \frac{\partial Q^*}{\partial d} = ?$

- $dy = (\frac{dy}{dx})dx$

  dy: the differential of y

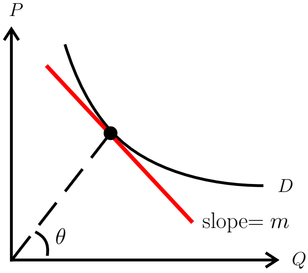
  dx: the differential of x

  dy/dx: the derivative of $y = f(x)$

$\Rightarrow$ $\frac{(dy)}{(dx)} = (\frac{dy}{dx}) \equiv f'(x)$

ex: $\epsilon^D \equiv \frac{dQ/Q}{dP/P} = (\frac{dQ}{dP})(\frac{P}{Q})$

$= \frac{1}{(dP/dQ)}(\frac{P}{Q}) = \frac{1}{m}\tan\theta$



- **Total Differentials**

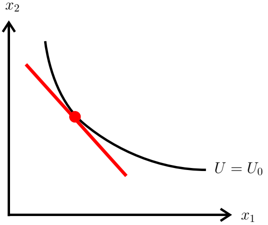
$$y = f(x_1, x_2) \quad \Rightarrow \quad dy = (\frac{\partial y}{\partial x_1})dx_1 + (\frac{\partial y}{\partial x_2})dx_2$$

$$\Rightarrow \quad \frac{\partial y}{\partial x_1} = \frac{dy}{dx_1}\Big|_{dx_2=0}$$

ex: $U = U(x_1, x_2) = U_0$ and $MU_1 = \frac{\partial U}{\partial x_1}, \quad MU_2 = \frac{\partial U}{\partial x_2}$

$\Rightarrow dU = MU_1 dx_1 + MU_2 dx_2 = 0$

$\Rightarrow \frac{dx_2}{dx_1} = -\frac{MU_1}{MU_2} = -MRS_{12}$



ex: $M = p_1 x_1 + p_2 x_2 + \cdots + p_n x_n$

$\Rightarrow dM = (p_1 dx_1 + x_1 dp_1) + (p_2 dx_2 + x_2 dp_2) + \cdots + (p_n dx_n + x_n dp_n)$

If $dp_1 = dp_2 = \cdots = dp_n = 0$, then

$$dM = p_1 dx_1 + p_2 dx_2 + \cdots + p_n dx_n$$

(i) if $dM = 0$, then $\frac{dx_2}{dx_1} = -\frac{p_1}{p_2}$

(ii) if $dM \neq 0$, then

$$\frac{dM}{M} = (\frac{p_1 x_1}{M})(\frac{dx_1}{x_1}) + (\frac{p_2 x_2}{M})(\frac{dx_2}{x_2}) + \cdots + (\frac{p_n x_n}{M})(\frac{dx_n}{x_n})$$

$$\Rightarrow S_1 \eta_1 + S_2 \eta_2 + \cdots + S_n \eta_n = 1$$

ex: $y = 5x_1^2 + 3x_2$

$$\Rightarrow dy = 10x_1\,dx_1 + 3\,dx_2$$

ex: $y = 3x_1^2 + x_1x_2^2$

$$\Rightarrow dy = (6x_1 + x_2^2)\,dx_1 + 2x_1x_2\,dx_2$$

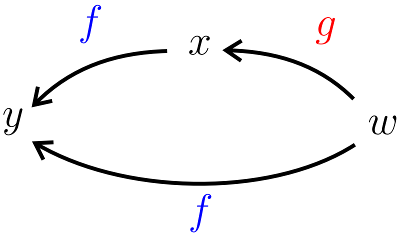
ex: $y = \frac{x_1 + x_2}{2x_1^2}$

$$\Rightarrow dy = \left[\frac{2x_1^2 - (x_1 + x_2)(4x_1)}{4x_1^4}\right] dx_1 + (\frac{1}{2x_1^2})\,dx_2$$

- **Total Derivatives**

Case 1:

$$y = f(x, w) = f(g(w), w)$$

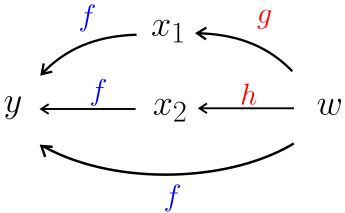


$$\Rightarrow \; dy = f_x \, dx + f_w \, dw = f_x \, g_w \, dw + f_w \, dw$$

$$\Rightarrow \; \frac{dy}{dw} = (\frac{\partial y}{\partial x})(\frac{dx}{dw}) + (\frac{\partial y}{\partial w})$$

Case 2:

$$y = f(x_1, x_2, w)$$
$$= f(g(w), h(w), w)$$

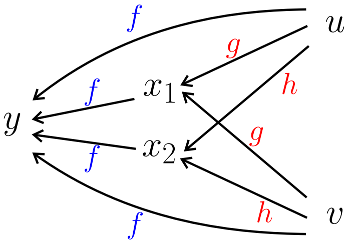


$$\Rightarrow dy = f_1\, dx_1 + f_2\, dx_2 + f_w\, dw$$
$$= f_1\, g_w\, dw + f_2\, h_w\, dw + f_w\, dw$$

$$\Rightarrow \frac{dy}{dw} = \left(\frac{\partial y}{\partial x_1}\right)\left(\frac{dx_1}{dw}\right) + \left(\frac{\partial y}{\partial x_2}\right)\left(\frac{dx_2}{dw}\right) + \left(\frac{\partial y}{\partial w}\right)$$

Case 3:

$$y = f(x_1, x_2, u, v)$$
$$= f(g(u,v), h(u,v), u, v)$$



$$\Rightarrow dy = f_1\,dx_1 + f_2\,dx_2 + f_u\,du + f_v\,dv$$
$$= f_1\,(g_u\,du + g_v\,dv) + f_2\,(h_u\,du + h_v\,dv) + f_u\,du + f_v\,dv$$
$$= (f_1\,g_u + f_2\,h_u + f_u)\,du + (f_1\,g_v + f_2\,h_v + f_v)\,dv$$
$$= (\frac{\S y}{\S u})\,du + (\frac{\S y}{\S v})\,dv$$

where $\frac{\S y}{\S u} \equiv \frac{dy}{du}\Big|_{dv=0} = (\frac{\partial y}{\partial x_1})(\frac{\partial x_1}{\partial u}) + (\frac{\partial y}{\partial x_2})(\frac{\partial x_2}{\partial u}) + (\frac{\partial y}{\partial u})$

is the partial total derivative of y with respect to u.

- **The Differential Version of Optimization Conditions**

$$y = f(x)$$

$$\Rightarrow \quad dy = f'(x)\,dx = 0$$

if and only if $\quad f'(x) = 0$ **[1st-order condition]**

$$\Rightarrow \quad d^2y = d(dy) = d(f'(x)\,dx)$$

$$= (df'(x))\,dx = (f''(x)\,dx)\,dx$$

$$= f''(x)(dx)^2 = f''(x)dx^2 > (<)0$$

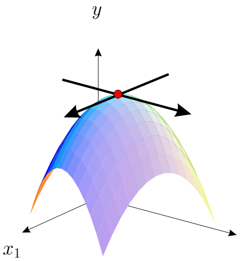
if and only if $\quad f''(x) > (<)0$ **[2nd-order condition]**

- **Two Variables Case**

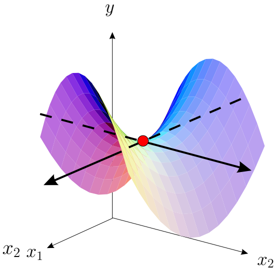
$y = f(x_1, x_2)$

$\Rightarrow \; dy = f_1\, dx_1 + f_2\, dx_2 = 0$ for arbitrary values of dx_1 and dx_2

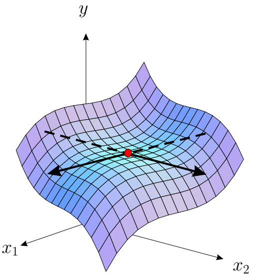
iff $f_1 = f_2 = 0$ **[1st-order condition]**



$y = -{x_1}^2 - {x_2}^2$



$y = {x_1}^2 - {x_2}^2$



$y = -{x_1}^3 - {x_2}^3$

ex: $y = f(x_1, x_2) = {x_1}^3 + 5x_1x_2 - {x_2}^2$

$$f_1(x_1, x_2) = 3{x_1}^2 + 5x_2 \overset{\text{set}}{=} 0$$
$$f_2(x_1, x_2) = 5x_1 - 2x_2 \overset{\text{set}}{=} 0$$

$\Rightarrow (x_1, x_2) = (0, 0)$ or $(-25/6, -125/12)$

- **2nd-Order Partial Derivatives**

Given $y = f(x_1, x_2)$ is a twice differentiable function, then

$$f_{11} \equiv \frac{\partial}{\partial x_1} f_1(x_1, x_2) = \frac{\partial}{\partial x_1}\left(\frac{\partial y}{\partial x_1}\right)$$
$$f_{12} \equiv \frac{\partial}{\partial x_2} f_1(x_1, x_2) = \frac{\partial}{\partial x_2}\left(\frac{\partial y}{\partial x_1}\right)$$

- **2nd-Order Condition**

$$d^2y \equiv d(dy) = (\frac{\partial}{\partial x_1}dy)dx_1 + (\frac{\partial}{\partial x_2}dy)dx_2$$

$$= [\, \frac{\partial}{\partial x_1}(f_1\, dx_1 + f_2\, dx_2)\,]dx_1 + [\, \frac{\partial}{\partial x_2}(f_1\, dx_1 + f_2\, dx_2)\,]dx_2$$

$$= (f_{11}\, dx_1 + f_{21}\, dx_2)dx_1 + (f_{12}\, dx_1 + f_{22}\, dx_2)dx_2$$

$$= f_{11}(dx_1)^2 + f_{21}(dx_2)(dx_1) + f_{12}(dx_1)(dx_2) + f_{22}(dx_2)^2$$

$$= \begin{bmatrix} dx_1 & dx_2 \end{bmatrix} \begin{bmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{bmatrix} \begin{bmatrix} dx_1 \\ dx_2 \end{bmatrix} \text{ (examples)}$$

$$= f_{11}\, {dx_1}^2 + 2f_{12}\, dx_1\, dx_2 + f_{22}\, {dx_2}^2 \qquad \text{[Young's Theorem]}$$

ex: $q = 5u^2 + 3uv + 2v^2$

$$\Rightarrow q = \begin{bmatrix} u & v \end{bmatrix} \begin{bmatrix} 5 & 3/2 \\ 3/2 & 2 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix}$$

ex: $z = -2x^2 + 2xy - y^2$

$$\Rightarrow z = \begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} -2 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

- **Young's Theorem**

  For a function

  $$y = f(x_1, x_2, \ldots, x_n)$$

  with continuous first- and second-order partial derivatives, the order of differentiation in computing the cross-partials is irrelevant. That is, $f_{ij} = f_{ji}$ for any pair i, j.

  $$f_{ij} \equiv \frac{\partial}{\partial x_j}\left(\frac{\partial f}{\partial x_i}\right) = \frac{\partial}{\partial x_i}\left(\frac{\partial f}{\partial x_j}\right) \equiv f_{ji}$$

$$d^2y = f_{11}\,{dx_1}^2 + 2f_{12}\,dx_1\,dx_2 + f_{22}\,{dx_2}^2$$
$$= f_{11}(dx_1 + \frac{f_{12}}{f_{11}}\,dx_2)^2 + \frac{f_{11}f_{22} - {f_{12}}^2}{f_{11}}(dx_2)^2$$

(1) $d^2y > 0$ iff $f_{11} > 0,\ f_{22} > 0,\ f_{11}\,f_{22} - {f_{12}}^2 > 0$

(2) $d^2y < 0$ iff $f_{11} < 0,\ f_{22} < 0,\ f_{11}\,f_{22} - {f_{12}}^2 > 0$

(3) If $f_{11}\,f_{22} - {f_{12}}^2 < 0$, then the point is a saddle point or an inflection point.

(examples)

- If the function $y = f(x_1, x_2)$ defined on $\mathbb{R}^2$ is twice continuously differentiable and

$$d^2y = f_{11}\, dx_1{}^2 + 2f_{12}\, dx_1\, dx_2 + f_{22}\, dx_2{}^2 > (<)\ 0$$

whenever at least one of dx_1 or dx_2 is nonzero, then $y = f(x_1, x_2)$ is a strictly convex (strictly concave) function.

- If the function $y = f(x_1, x_2)$ defined on $\mathbb{R}^2$ is twice continuously differentiable, then it is convex (concave) if and only if

$$d^2y = f_{11}\, dx_1{}^2 + 2f_{12}\, dx_1\, dx_2 + f_{22}\, dx_2{}^2 \geq (\leq)\ 0$$

- **Three Variables Case**

$y = f(x_1, x_2, x_3)$

(1) $dy = f_1\, dx_1 + f_2\, dx_2 + f_3\, dx_3$

$\Rightarrow dy = 0 \quad \text{iff} \quad f_1 = f_2 = f_3 = 0$ **[1st-order condition]**

(2)
$$\begin{aligned} d^2y = {} & (f_{11}\, dx_1 + f_{12}\, dx_2 + f_{13}\, dx_3)dx_1 \\ & +(f_{21}\, dx_1 + f_{22}\, dx_2 + f_{23}\, dx_3)dx_2 \\ & +(f_{31}\, dx_1 + f_{32}\, dx_2 + f_{33}\, dx_3)dx_3 \\ = {} & \begin{bmatrix} dx_1 & dx_2 & dx_3 \end{bmatrix} \begin{bmatrix} f_{11} & f_{12} & f_{13} \\ f_{21} & f_{22} & f_{23} \\ f_{31} & f_{32} & f_{33} \end{bmatrix} \begin{bmatrix} dx_1 \\ dx_2 \\ dx_3 \end{bmatrix} \end{aligned}$$

- Let H be the **Hessian** Matrix associated with a twice continuously differentiable function $y = f(\mathbf{x}),\ \mathbf{x} \in \mathbb{R}^n$.

$$H = \begin{bmatrix} f_{11} & f_{12} & \cdots & f_{1n} \\ f_{21} & f_{22} & \cdots & f_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ f_{n1} & f_{n2} & \cdots & f_{nn} \end{bmatrix}$$

- Denote $|H_1|, |H_2|, \cdots, |H_n|$ as the leading principal minors:

$$|H_1| = |f_{11}|,\ |H_2| = \begin{vmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{vmatrix},\ |H_3| = \begin{vmatrix} f_{11} & f_{12} & f_{13} \\ f_{21} & f_{22} & f_{23} \\ f_{31} & f_{32} & f_{33} \end{vmatrix}$$

$$
\begin{aligned}
d^2y &= \sum_{i=1}^{3}\sum_{j=1}^{3}(f_{ij}\,dx_i\,dx_j) \\
&= f_{11}(dx_1 + \frac{f_{12}}{f_{11}}\,dx_2 + \frac{f_{13}}{f_{11}}\,dx_3)^2 \\
&\qquad +(f_{22} - \frac{{f_{12}}^2}{f_{11}})(dx_2)^2 + (f_{33} - \frac{{f_{13}}^2}{f_{11}})(dx_3)^2 \\
&\qquad\qquad +2(\frac{f_{11}f_{23} - f_{12}f_{13}}{f_{11}})(dx_2)(dx_3) \\
&= |H_1|(dx_1 + \frac{f_{12}}{f_{11}}\,dx_2 + \frac{f_{13}}{f_{11}}\,dx_3)^2 \\
&\qquad +\frac{|H_2|}{|H_1|}(dx_2 + \frac{f_{11}f_{23} - f_{12}f_{13}}{f_{11}f_{22} - f_{12}^2}\,dx_3)^2 + \frac{|H_3|}{|H_2|}(dx_3)^2
\end{aligned}
$$

- $d^2y > 0$ iff $|H_1| > 0,\ |H_2| > 0,\ |H_3| > 0$
- $d^2y < 0$ iff $|H_1| < 0,\ |H_2| > 0,\ |H_3| < 0$

- **n-Variables Case**

(1) $d^2y > 0$ iff $|H_1| > 0,\ |H_2| > 0,\ |H_3| > 0, \cdots,\ |H_n| > 0$

$+ \quad + \quad + \quad \cdots \quad +$

and H is said to be a **positive definite** matrix.

(2) $d^2y < 0$ iff $|H_1| < 0,\ |H_2| > 0,\ |H_3| < 0,\ |H_4| > 0, \cdots$

$- \quad + \quad - \quad + \quad \cdots$

and H is said to be a **negative definite** matrix.

(3) $d^2y \geq 0$ iff $|H_1| \geq 0,\ |H_2| \geq 0,\ |H_3| \geq 0, \cdots,\ |H_n| \geq 0$

and H is said to be a **positive semidefinite** matrix.

(4) $d^2y \leq 0$ iff $|H_1| \leq 0,\ |H_2| \geq 0,\ |H_3| \leq 0,\ |H_4| \geq 0, \cdots$

and H is said to be a **negative semidefinite** matrix.

ex: $y = f(x_1, x_2, x_3) = 3{x_1}^2 - 2x_1x_2 + 4x_1x_3 + 5{x_2}^2 + 4{x_3}^2 - 2x_2x_3$

$\Rightarrow$ $f_1(x_1, x_2, x_3) = 6x_1 - 2x_2 + 4x_3$

$$f_2(x_1, x_2, x_3) = -2x_1 + 10x_2 - 2x_3 \quad \Rightarrow \quad H = \begin{bmatrix} 6 & -2 & 4 \\ -2 & 10 & -2 \\ 4 & -2 & 8 \end{bmatrix}$$

$f_3(x_1, x_2, x_3) = 4x_1 + 8x_3 - 2x_2$

$$\Rightarrow \; |H_1| = 6 > 0, \qquad |H_2| = \begin{vmatrix} 6 & -2 \\ -2 & 10 \end{vmatrix} = 56 > 0,$$

$$|H_3| = \begin{vmatrix} 6 & -2 & 4 \\ -2 & 10 & -2 \\ 4 & -2 & 8 \end{vmatrix} = 296 > 0,$$

$\Rightarrow$ H is a positive definite matrix.

ex: $y = f(x_1, x_2, x_3) = 2{x_1}^2 + 3{x_2}^2 - {x_3}^2 + 6x_1x_2 - 8x_1x_3 - 2x_2x_3$

$$\Rightarrow H = \begin{bmatrix} 4 & 6 & -8 \\ 6 & 6 & -2 \\ -8 & -2 & -2 \end{bmatrix}$$

$$\Rightarrow |H_1| = 4 > 0, \qquad |H_2| = \begin{vmatrix} 4 & 6 \\ 6 & 6 \end{vmatrix} = -12 < 0,$$

$\Rightarrow$ H is neither positive nor negative definite.

ex: Suppose that a monopolistic firm sells a single product in three separate markets and the demands facing this firm are as follows:

$$P_1 = 63 - 4Q_1, \qquad P_2 = 105 - 5Q_2, \qquad P_3 = 75 - 6Q_3$$

and that the total-cost function is

$$C = 20 + 15Q.$$

Please solve the profit maximization problem for this firm.

- Note that $R_i = P_i Q_i$, hence

$$\begin{aligned} \frac{d}{dQ_i} R_i &= P_i + \left(\frac{dP_i}{dQ_i} \right) Q_i \\ &= P_i \left[1 + \left(\frac{dQ_i}{dP_i} \frac{P_i}{Q_i} \right)^{-1} \right] = P_i \left(1 - \frac{1}{|\epsilon_i|} \right) \end{aligned}$$

$$\begin{aligned}\pi &= R_1 + R_2 + R_3 - C \\ &= (63 - 4Q_1)Q_1 + (105 - 5Q_2)Q_2 + (75 - 6Q_3)Q_3 \\ &\qquad -[20 + 15(Q_1 + Q_2 + Q_3)] \\ &= -20 + 48Q_1 - 4{Q_1}^2 + 90Q_2 - 5{Q_2}^2 + 60Q_3 - 6{Q_3}^2\end{aligned}$$

$$\Rightarrow \begin{aligned}\pi_1 &= 48 - 8Q_1 \overset{\text{set}}{=} 0 \\ \pi_2 &= 90 - 10Q_2 \overset{\text{set}}{=} 0 \quad \Rightarrow \quad (\overline{Q_1}, \overline{Q_2}, \overline{Q_3}) = (6, 9, 5) \\ \pi_3 &= 60 - 12Q_3 \overset{\text{set}}{=} 0\end{aligned}$$

$$\Rightarrow H = \begin{bmatrix} -8 & 0 & 0 \\ 0 & -10 & 0 \\ 0 & 0 & -12 \end{bmatrix} \text{ is negative definite.}$$

Thus, the equilibrium profit is a maximum.

- **Eigenvalue and Eigenvector**

  Given an $n \times n$ matrix A, we can find a scalar λ and an $n \times 1$ vector $\mathbf{x} \neq \mathbf{0}_{n\times 1}$ such that

$$A\mathbf{x} = \lambda\mathbf{x},$$

  where λ is an **eigenvalue** (characteristic root) of A
  and $\mathbf{x}$ is an **eigenvector** (characteristic vector) of A.

- $A\mathbf{x} = \lambda\mathbf{x} \quad \Rightarrow \quad (A - \lambda I)\mathbf{x} = \mathbf{0}_{n\times 1}$

- If $\mathbf{x}$ is required not to be a trivial solution (i.e., $\mathbf{x} \neq \mathbf{0}$),

$\Rightarrow$ $|A - \lambda I| = 0$ i.e., $(A - \lambda I)$ is singular.

ex: $A = \begin{bmatrix} 4 & 1 \\ 2 & 3 \end{bmatrix}$

$\Rightarrow |A - \lambda I| = \begin{vmatrix} 4-\lambda & 1 \\ 2 & 3-\lambda \end{vmatrix} = \lambda^2 - 7\lambda + 10 = 0$

$\Rightarrow \lambda_1 = 2$ and $\lambda_2 = 5$

$\Rightarrow \begin{bmatrix} 4-2 & 1 \\ 2 & 3-2 \end{bmatrix} \mathbf{x}_1 = \begin{bmatrix} 2 & 1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} a_1 \\ b_1 \end{bmatrix} = \mathbf{0}$

and $\begin{bmatrix} 4-5 & 1 \\ 2 & 3-5 \end{bmatrix} \mathbf{x}_2 = \begin{bmatrix} -1 & 1 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} a_2 \\ b_2 \end{bmatrix} = \mathbf{0}$

$\Rightarrow$ By **normalization** (Let $\|\mathbf{x}\| = \sqrt{\mathbf{x}^T\mathbf{x}} = \sqrt{a^2 + b^2} = 1$)

$\mathbf{x}_1 = \begin{bmatrix} 1/\sqrt{5} \\ -2/\sqrt{5} \end{bmatrix}$ and $\mathbf{x}_2 = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$

ex: $\begin{bmatrix} 2 & 4 \\ 1 & 2 \end{bmatrix}$

ex: $A = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix} \Rightarrow |A - \lambda I| = \begin{vmatrix} 2-\lambda & 1 & 1 \\ 1 & 2-\lambda & 1 \\ 1 & 1 & 2-\lambda \end{vmatrix}$

$$= -\lambda^3 + 6\lambda^2 - 9\lambda + 4 = 0$$

$\Rightarrow \lambda_1 = 1, \;\; \lambda_2 = 1, \;\; \lambda_3 = 4$

(i) $\begin{bmatrix} 2-1 & 1 & 1 \\ 1 & 2-1 & 1 \\ 1 & 1 & 2-1 \end{bmatrix} \mathbf{x}_1 = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} a_1 \\ b_1 \\ c_1 \end{bmatrix} = \mathbf{0}$

$\Rightarrow a_1 + b_1 + c_1 = 0$ and (by normalization) ${a_1}^2 + {b_1}^2 + {c_1}^2 = 1$

$\Rightarrow \mathbf{x}_1 = \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \\ 0 \end{bmatrix}$ and $\mathbf{x}_2 = \begin{bmatrix} 1/\sqrt{6} \\ 1/\sqrt{6} \\ -2/\sqrt{6} \end{bmatrix}$

(ii) $$\begin{bmatrix} 2-4 & 1 & 1 \\ 1 & 2-4 & 1 \\ 1 & 1 & 2-4 \end{bmatrix} \mathbf{x}_3 = \begin{bmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{bmatrix} \begin{bmatrix} a_3 \\ b_3 \\ c_3 \end{bmatrix} = \mathbf{0}$$

$\Rightarrow$ $a_3 = b_3 = c_3$ and (by normalization) ${a_3}^2 + {b_3}^2 + {c_3}^2 = 1$

$$\Rightarrow \mathbf{x}_3 = \begin{bmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix}$$

ex: $$\begin{bmatrix} 1 & -1 & -1 \\ 1 & -1 & 0 \\ 1 & 0 & -1 \end{bmatrix}$$

- $|A - \lambda I|$

$$= \begin{vmatrix} a_{11} - \lambda & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} - \lambda & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} - \lambda \end{vmatrix}$$

(is an nth-degree polynomial in λ)

$$= (-1)^n[\lambda^n - \alpha_1\lambda^{n-1} + \alpha_2\lambda^{n-2} + \cdots + (-1)^{n-1}\alpha_{n-1}\lambda + (-1)^n\alpha_n]$$

(and thus has n solutions $\lambda_1, \lambda_2, \cdots, \lambda_n$)

$$= (-1)^n(\lambda - \lambda_1)(\lambda - \lambda_2)\cdots(\lambda - \lambda_n)$$

- Note that α_1 denotes the sum and α_n the product of all eigenvalues.

- If $\lambda = 0$, then $|A| = \alpha_n = \lambda_1 \lambda_2 \cdots \lambda_n$

  (1) The determinant of A equals the product of all its eigenvalues.

  (2) A is nonsingular if and only if no eigenvalue equals zero.

- $\alpha_1 = \lambda_1 + \lambda_2 + \cdots + \lambda_n$

  $= a_{11} + a_{22} + \cdots + a_{nn} \equiv \text{trace}(A)$

  (3) The sum of all the eigenvalues of A equals the trace of A.

ex: $\begin{bmatrix} 4 & 1 \\ 2 & 3 \end{bmatrix}$

$\Rightarrow \lambda_1 = 2$ and $\lambda_2 = 5$

ex: $\begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}$

$\Rightarrow \lambda_1 = 1, \ \lambda_2 = 1, \ \lambda_3 = 4$

- $|A - \lambda I| = |(A - \lambda I)^T| = |A^T - \lambda I|$
  (4) A^T has the same eigenvalues as A's.

- $|A - \lambda I| = k^{-n}|k(A - \lambda I)| = k^{-n}|kA - (k\lambda)I|$
  (5) The eigenvalues of kA equals k-folds the eigenvalues of A.

- If A^{-1} exists, then
$$\begin{aligned}|A - \lambda I| &= |A - \lambda A A^{-1}| \\ &= |(-\lambda A)(-\frac{1}{\lambda} I + A^{-1})| \\ &= (-\lambda)^n |A||A^{-1} - \frac{1}{\lambda} I|\end{aligned}$$
  (6) The eigenvalues of A^{-1} are the reciprocal of the eigenvalues of A.

- **Theorem**

  If A is a **symmetric** matrix with **all real** elements, then the n eigenvalues are all real numbers.

- **Theorem** (**important!!**)

  For a real **symmetric** matrix A,

$$\mathbf{x}_i{}^T\mathbf{x}_i = 1 \text{ and } \mathbf{x}_i{}^T\mathbf{x}_j = 0, \quad \forall\, i \neq j$$

(normalization) (orthogonal)

$\Rightarrow$ $(\mathbf{x}_1, \mathbf{x}_2, \cdots, \mathbf{x}_n)$ are said to be a set of **orthonormal** vectors.

Proof $\mathbf{x}_i{}^T\lambda_j\mathbf{x}_j = \mathbf{x}_i{}^T A\mathbf{x}_j = (\mathbf{x}_i{}^T A\mathbf{x}_j)^T$

$$= \mathbf{x}_j{}^T A^T(\mathbf{x}_i{}^T)^T = \mathbf{x}_j{}^T A\mathbf{x}_i = \mathbf{x}_j{}^T\lambda_i\mathbf{x}_i$$

$\Rightarrow$ $\lambda_j(\mathbf{x}_i{}^T\mathbf{x}_j) = \lambda_i(\mathbf{x}_j{}^T\mathbf{x}_i)$ or $(\lambda_j - \lambda_i)(\mathbf{x}_i{}^T\mathbf{x}_j) = 0$

$\Rightarrow$ If $\lambda_j \neq \lambda_i$, then $\mathbf{x}_i{}^T\mathbf{x}_j = 0$.

If $\lambda_j = \lambda_i$, then we can find $\mathbf{x}_i$ and $\mathbf{x}_j$ such that $\mathbf{x}_i{}^T\mathbf{x}_j = 0$.

- $d^2q = \begin{bmatrix} dx_1 & \cdots & dx_n \end{bmatrix} \begin{bmatrix} f_{11} & \cdots & f_{1n} \\ \vdots & \ddots & \vdots \\ f_{n1} & \cdots & f_{nn} \end{bmatrix} \begin{bmatrix} dx_1 \\ \vdots \\ dx_n \end{bmatrix} = \mathbf{u}^T H \mathbf{u}$

  Let $B = \begin{bmatrix} | & | & & | \\ \mathbf{x}_1 & \mathbf{x}_2 & \cdots & \mathbf{x}_n \\ | & | & & | \end{bmatrix}_{n\times n}$

$\Rightarrow$ B is nonsingular (WHY?) and hence B^{-1} exists

Let $\mathbf{y} = B^{-1}\mathbf{u}$ (or $\mathbf{u} = B\mathbf{y}$)

$$\Rightarrow d^2q = \mathbf{u}^T H \mathbf{u} = (B\mathbf{y})^T H (B\mathbf{y}) = \mathbf{y}^T (B^T H B)\mathbf{y}$$

$$= \mathbf{y}^T \begin{bmatrix} - & {\mathbf{x}_1}^T & - \\ - & {\mathbf{x}_2}^T & - \\ - & \vdots & - \\ - & {\mathbf{x}_n}^T & - \end{bmatrix} \begin{bmatrix} | & | & & | \\ \lambda_1 \mathbf{x}_1 & \lambda_2 \mathbf{x}_2 & \cdots & \lambda_n \mathbf{x}_n \\ | & | & & | \end{bmatrix} \mathbf{y}$$

$$= \mathbf{y}^T \begin{bmatrix} \lambda_1 {\mathbf{x}_1}^T \mathbf{x}_1 & \lambda_2 {\mathbf{x}_1}^T \mathbf{x}_2 & \cdots & \lambda_n {\mathbf{x}_1}^T \mathbf{x}_n \\ \lambda_1 {\mathbf{x}_2}^T \mathbf{x}_1 & \lambda_2 {\mathbf{x}_2}^T \mathbf{x}_2 & \cdots & \lambda_n {\mathbf{x}_2}^T \mathbf{x}_n \\ \vdots & \vdots & \ddots & \vdots \\ \lambda_1 {\mathbf{x}_n}^T \mathbf{x}_1 & \lambda_2 {\mathbf{x}_n}^T \mathbf{x}_2 & \cdots & \lambda_n {\mathbf{x}_n}^T \mathbf{x}_n \end{bmatrix} \mathbf{y}$$

$$= \mathbf{y}^T \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix} \mathbf{y} = \lambda_1 {y_1}^2 + \lambda_2 {y_2}^2 + \cdots + \lambda_n {y_n}^2$$

- **Conclusions**

1. H is positive definite if and only if $\lambda_i > 0 \ \ \forall\, i$
2. H is negative definite if and only if $\lambda_i < 0 \ \ \forall\, i$
3. H is positive semidefinite if and only if $\lambda_i \geq 0 \ \ \forall\, i$
4. H is negative semidefinite if and only if $\lambda_i \leq 0 \ \ \forall\, i$
5. H is indefinite if and only if some λs are positive while others are negative.

ex: Find the extreme value(s) of $q = -1.5x^2 + 3xz + 2y - y^2 - 3z^2$ and determine whether they are maxima or minima with the eigenvalue test.

$$\Rightarrow \; q_x = -3x + 3z \underset{\text{set}}{=} 0$$

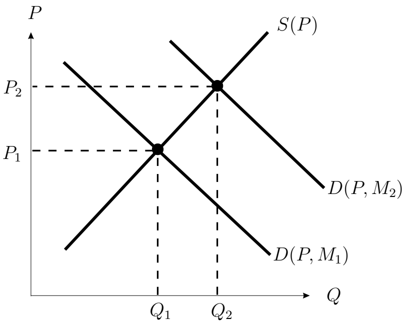
$$q_y = 2 - 2y \underset{\text{set}}{=} 0 \qquad \Rightarrow \quad (\overline{x}, \overline{y}, \overline{z}) = (0, 1, 0)$$

$$q_z = 3x - 6z \underset{\text{set}}{=} 0$$

$$\Rightarrow \; H = \begin{bmatrix} -3 & 0 & 3 \\ 0 & -2 & 0 \\ 3 & 0 & -6 \end{bmatrix} \Rightarrow \begin{vmatrix} -3-\lambda & 0 & 3 \\ 0 & -2-\lambda & 0 \\ 3 & 0 & -6-\lambda \end{vmatrix}$$

$$= -(\lambda + 2)(\lambda^2 + 9\lambda + 9) = 0$$

$$\Rightarrow \; \lambda_1 = -2, \quad \lambda_2 = \frac{-9 + 3\sqrt{5}}{2}, \quad \lambda_3 = \frac{-9 - 3\sqrt{5}}{2}$$



At each equilibrium point,

$$Z(P, M) = D(P, M) - S(P) = 0$$

Q: $P = P(M)$ **?**

Q: If yes, what will $\frac{dP}{dM}$ be **?**

ex: $y = f(x) = 2x^2$

$\Rightarrow F(y, x) = y - 2x^2 = 0$

ex: $y = f(x_1, x_2) = \dfrac{x_1}{x_1 + x_2^2}$

$\Rightarrow F(y, x_1, x_2) = y(x_1 + x_2^2) - x_1 = 0$

Q: Does there exist a function $f : \mathbb{R}^m \to \mathbb{R}$ (i.e., $y = f(\mathbf{x})$, $\mathbf{x} \in \mathbb{R}^m$) corresponding to the relationship defined by $F : \mathbb{R}^{m+1} \to \mathbb{R}$ (i.e., $F(y, \mathbf{x}) = 0$) **?**

- **Implicit Function Theorem**

If (1) $F:\mathbb{R}^{m+1}\to\mathbb{R}$,

(2) all the first partial derivatives of F are continuous,

(3) $\dfrac{\partial F(y,\mathbf{x})}{\partial y}\neq 0$, at the point $(\bar{y},\bar{\mathbf{x}})$ satisfying $F(y,\mathbf{x})=0$,

then there exist $N_{\epsilon_1}(\bar{\mathbf{x}})$ and $N_{\epsilon_2}(\bar{y})$

and a function $f:N_{\epsilon_1}(\bar{\mathbf{x}})\to N_{\epsilon_2}(\bar{y})$ satisfying

$$F(f(\mathbf{x}),\mathbf{x})=0,\quad \forall\,\mathbf{x}\in N_{\epsilon_1}(\bar{\mathbf{x}})$$

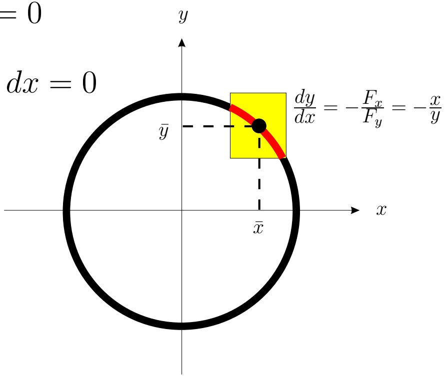
Also, f and $f_i,\ i=1\sim m$ are continuous.

ex: $F(y,x) = x^2 + y^2 - 1 = 0$

$\Rightarrow$ $F_y = 2y, \quad F_x = 2x$ are continuous

$\Rightarrow$ $F_y \neq 0$ except when $y = 0$

$$F_y\,dy + F_x\,dx = 2y\,dy + 2x\,dx = 0$$

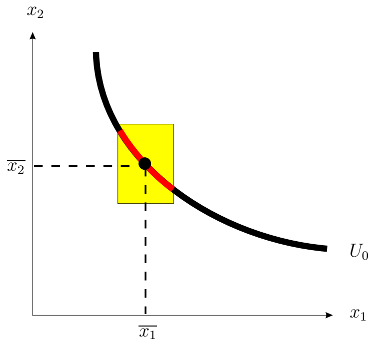


ex: $U = U(x_1, x_2) = U_0$

$$\Rightarrow\ U(x_1, f(x_1)) = U_0, \quad \forall\ x_1 \in N_\epsilon(\bar{x_1})$$

$$\Rightarrow\ U_1(x_1, f(x_1))\, dx_1 + U_2(x_1, f(x_1)) f'(x_1)\, dx_1 = 0$$

$$\Rightarrow\ f'(x_1) = -\frac{U_1(x_1, f(x_1))}{U_2(x_1, f(x_1))} = -MRS_{12}.$$



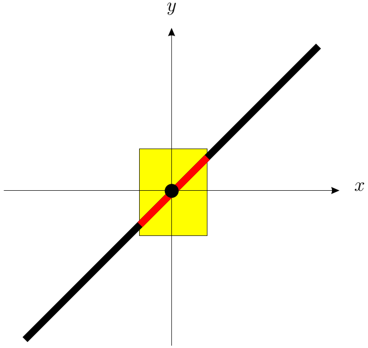
- Note that the implicit function theorem is sufficient but not necessary.

ex: $F(y, x) = (x - y)^3 = 0$

$\Rightarrow$ $F_x = 3x^2 - 6xy + 3y^2$

$F_y = -3x^2 + 6xy - 3y^2$

$\Rightarrow$ $F_y(0, 0) = 0$



- **Implicit Function Rule**

$$F(y, \mathbf{x}) = 0 \quad \text{with} \quad F_y \neq 0$$

$$\Rightarrow \; F_y\,dy + F_1\,dx_1 + F_2\,dx_2 + \cdots + F_m\,dx_m = 0$$

and $dy = f_1\,dx_1 + f_2\,dx_2 + \cdots + f_m\,dx_m \quad [\ \because y = f(\mathbf{x})\ ]$

$$\Rightarrow \; (F_y f_1 + F_1)\,dx_1 + (F_y f_2 + F_2)\,dx_2 + \cdots + (F_y f_m + F_m)\,dx_m = 0$$

$$\Rightarrow \; F_y f_i + F_i = 0, \quad \forall\ i$$

$$\Rightarrow \; f_i \equiv \frac{\partial y}{\partial x_i} = -\frac{F_i}{F_y}, \quad \forall\ i$$

ex: $Z(P, M) = D(P, M) - S(P) = 0$

$$\Rightarrow \frac{\partial Z(P, M)}{\partial M} = \frac{\partial D(P, M)}{\partial M} > 0$$

$$\frac{\partial Z(P, M)}{\partial P} = \frac{\partial D(P, M)}{\partial P} - \frac{dS(P)}{dP} < 0$$

$$\Rightarrow P = P(M)$$

$$\frac{dP}{dM} = -\frac{\partial Z/\partial M}{\partial Z/\partial P} > 0 \quad \text{and} \quad \frac{dQ}{dM} = \left(\frac{dS}{dP}\right)\left(\frac{dP}{dM}\right) > 0$$

ex: $x^2 + y^2 + z^2 = 1$

- **Implicit Function Theorem (Extension)**

Given $F^i(\mathbf{y}, \mathbf{x}) = 0,\ i = 1 \sim n,\ \mathbf{y} \in \mathbb{R}^n,\ \mathbf{x} \in \mathbb{R}^m$. If

(1) function $F^1, F^2, \cdots, F^n$ all have continuous first partial derivatives with respect to all the $\mathbf{y}$ and $\mathbf{x}$ variables.

(2) at the point $(\mathbf{y}, \mathbf{x})$ satisfying $F^i(\mathbf{y}, \mathbf{x}) = 0,\ i = 1 \sim n$,

$$|J| \equiv \left| \frac{\partial(F^1, F^2, \cdots, F^n)}{\partial(y_1, y_2, \cdots, y_n)} \right| = \begin{vmatrix} \frac{\partial F^1}{\partial y_1} & \frac{\partial F^1}{\partial y_2} & \cdots & \frac{\partial F^1}{\partial y_n} \\ \frac{\partial F^2}{\partial y_1} & \frac{\partial F^2}{\partial y_2} & \cdots & \frac{\partial F^2}{\partial y_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial F^n}{\partial y_1} & \frac{\partial F^n}{\partial y_2} & \cdots & \frac{\partial F^n}{\partial y_n} \end{vmatrix} \neq 0,$$

then there exist an m-dimensional neighborhood $N_\epsilon(\bar{\mathbf{x}})$ in which all $y_j,\ j = 1 \sim n,$ are functions of $\mathbf{x}$.

ex: Given $x^2 + y^2 + z^2 = 3$ and $x + 2y + 3z = 0$, are x and y defined as functions of z around the point $(x = 1,\ y = 1,\ z = -1)$ **?**

$$\Rightarrow\ F^1(x, y, z) = x^2 + y^2 + z^2 - 3 = 0$$
$$F^2(x, y, z) = x + 2y + 3z = 0$$

$$\Rightarrow\ |J| = \begin{vmatrix} \frac{\partial F^1}{\partial x} & \frac{\partial F^1}{\partial y} \\ \frac{\partial F^2}{\partial x} & \frac{\partial F^2}{\partial y} \end{vmatrix} = \begin{vmatrix} 2x & 2y \\ 1 & 2 \end{vmatrix} = 4x - 2y$$

which equals 2 at $(x = 1,\ y = 1,\ z = -1)$

$\Rightarrow$ Thus, $x = x(z)$ and $y = y(z)$ around $(1, 1, -1)$

ex: $$Y = C + I_0 + G_0$$
$$C = \alpha + \beta(Y - T)$$
$$T = \gamma + \delta Y$$

$$\Rightarrow F^1(Y, C, T, I_0, G_0, \alpha, \beta, \gamma, \delta) = Y - C - I_0 - G_0 = 0$$
$$F^2(Y, C, T, I_0, G_0, \alpha, \beta, \gamma, \delta) = C - \alpha - \beta(Y - T) = 0$$
$$F^3(Y, C, T, I_0, G_0, \alpha, \beta, \gamma, \delta) = T - \gamma - \delta Y = 0$$

$$\Rightarrow |J| = \begin{vmatrix} \frac{\partial F^1}{\partial Y} & \frac{\partial F^1}{\partial C} & \frac{\partial F^1}{\partial T} \\ \frac{\partial F^2}{\partial Y} & \frac{\partial F^2}{\partial C} & \frac{\partial F^2}{\partial T} \\ \frac{\partial F^3}{\partial Y} & \frac{\partial F^3}{\partial C} & \frac{\partial F^3}{\partial T} \end{vmatrix} = \begin{vmatrix} 1 & -1 & 0 \\ -\beta & 1 & \beta \\ -\delta & 0 & 1 \end{vmatrix} = 1 + \beta\delta - \beta \neq 0$$

$$\Rightarrow Y = Y(I_0, G_0, \alpha, \beta, \gamma, \delta)$$
$$C = C(I_0, G_0, \alpha, \beta, \gamma, \delta)$$
$$T = T(I_0, G_0, \alpha, \beta, \gamma, \delta)$$

- **Implicit Function Rule (Extension)**

$$F^i = 0 \quad \Rightarrow \quad dF^i = 0, \ \forall \ i$$

$$\Rightarrow \ \frac{\partial F^i}{\partial y_1} dy_1 + \cdots + \frac{\partial F^i}{\partial y_n} dy_n = -(\frac{\partial F^i}{\partial x_1} dx_1 + \cdots + \frac{\partial F^i}{\partial x_m} dx_m), \ \forall \ i$$

Let $dx_k = 0, \ \forall k \neq 1$, then

$$\begin{bmatrix} \frac{\partial F^1}{\partial y_1} & \cdots & \frac{\partial F^1}{\partial y_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial F^n}{\partial y_1} & \cdots & \frac{\partial F^n}{\partial y_n} \end{bmatrix} \begin{bmatrix} \frac{\partial y_1}{\partial x_1} \\ \vdots \\ \frac{\partial y_n}{\partial x_1} \end{bmatrix} = \begin{bmatrix} -\frac{\partial F^1}{\partial x_1} \\ \vdots \\ -\frac{\partial F^n}{\partial x_1} \end{bmatrix}$$

$\Rightarrow \ \frac{\partial y_j}{\partial x_1} = \frac{|J_j|}{|J|}, \ j = 1 \sim n$ and $|J| \neq 0$ guarantees a unique solution.

ex: $F^1(x,y,z) = x^2 + y^2 + z^2 - 3 = 0$

$F^2(x,y,z) = x + 2y + 3z = 0$

$$\Rightarrow \begin{array}{ccccc} 2x\ dx & + & 2y\ dy & = & -2z\ dz \\ 1\ dx & + & 2\ dy & = & -3\ dz \end{array}$$

$$\Rightarrow \begin{bmatrix} 2x & 2y \\ 1 & 2 \end{bmatrix} \begin{bmatrix} dx/dz \\ dy/dz \end{bmatrix} = \begin{bmatrix} -2z \\ -3 \end{bmatrix}$$

$$\Rightarrow \frac{dx}{dz} = \frac{\begin{vmatrix} -2z & 2y \\ -3 & 2 \end{vmatrix}}{\begin{vmatrix} 2x & 2y \\ 1 & 2 \end{vmatrix}} = \frac{6y - 4z}{4x - 2y}$$

which equals 5 at $(1, 1, -1)$.

ex: $F^1 = Y - C - I_0 - G_0 = 0$
$F^2 = C - \alpha - \beta(Y - T) = 0$
$F^3 = T - \gamma - \delta Y = 0$

$$\Rightarrow \begin{array}{rcrcrcrcr} dY & - & dC & & & = & dI_0 & + & dG_0 \\ -\beta\, dY & + & dC & + & \beta\, dT & = & d\alpha & + & (Y-T)d\beta \\ -\delta\, dY & + & & & dT & = & d\gamma & + & Y d\delta \end{array}$$

Let $dI_0 = dG_0 = d\alpha = d\beta = d\gamma = 0$, then

$$\begin{bmatrix} 1 & -1 & 0 \\ -\beta & 1 & \beta \\ -\delta & 0 & 1 \end{bmatrix} \begin{bmatrix} \partial Y/\partial\delta \\ \partial C/\partial\delta \\ \partial T/\partial\delta \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ Y \end{bmatrix}$$

$$\Rightarrow \frac{\partial Y}{\partial \delta} = \frac{1}{1+\beta\delta-\beta} \begin{vmatrix} 0 & -1 & 0 \\ 0 & 1 & \beta \\ Y & 0 & 1 \end{vmatrix} = \frac{-\beta Y}{1+\beta\delta-\beta}$$

which equals $\dfrac{-\beta\bar{Y}}{1+\beta\delta-\beta}$ at $(\bar{Y}, \bar{C}, \bar{T})$.

Constrained Optimization

refer to textbook

Ch.12 Optimization with Equality Constraints

- max $U(x_1, x_2) = x_1 x_2 + 2x_1$

  s.t. $4x_1 + 2x_2 = 60$

Way 1 :

$$x_2 = 30 - 2x_1$$

$$\Rightarrow U = x_1(30 - 2x_1) + 2x_1 = -2{x_1}^2 + 32x_1$$

$$\Rightarrow \frac{dU}{dx_1} = -4x_1 + 32 \overset{\text{set}}{=} 0$$ **[1st-order condition]**

$$\Rightarrow \overline{x}_1 = 8\ , \quad \overline{x}_2 = 14$$

- max $U(x_1, x_2) = x_1 x_2 + 2x_1$

  s.t. $4x_1 + 2x_2 = 60$

Way 2 (Lagrange-Multiplier Method):

$$\mathcal{L}(x_1, x_2, \lambda) = (x_1 x_2 + 2x_1) + \lambda(60 - 4x_1 - 2x_2)$$

$$\Rightarrow \left.\begin{aligned} \mathcal{L}_\lambda &= 60 - 4x_1 - 2x_2 \overset{\text{set}}{=} 0 \\ \mathcal{L}_1 &= x_2 + 2 - 4\lambda \overset{\text{set}}{=} 0 \\ \mathcal{L}_2 &= x_1 - 2\lambda \overset{\text{set}}{=} 0 \end{aligned}\right\}$$

[1st-order conditions]

$$\Rightarrow \quad \overline{x}_1 = 8 \ , \quad \overline{x}_2 = 14$$

- max $U = x^2 + 2xy + yw^2$

  s.t. $2x + y + w^2 = 24$

  $x + w = 8$

$$\Rightarrow \mathcal{L} = x^2 + 2xy + yw^2 + \lambda_1(24 - 2x - y - w^2) + \lambda_2(8 - x - w)$$

$$\Rightarrow \left.\begin{aligned} \mathcal{L}_{\lambda 1} &= 24 - 2x - y - w^2 \overset{\text{set}}{=} 0 \\ \mathcal{L}_{\lambda 2} &= 8 - x - w \overset{\text{set}}{=} 0 \\ \mathcal{L}_x &= 2x + 2y - 2\lambda_1 - \lambda_2 \overset{\text{set}}{=} 0 \\ \mathcal{L}_y &= 2x + w^2 - \lambda_1 \overset{\text{set}}{=} 0 \\ \mathcal{L}_w &= 2yw - 2\lambda_1 w - \lambda_2 \overset{\text{set}}{=} 0 \end{aligned}\right\} \quad \textbf{[1st-order conditions]}$$

$$\Rightarrow \overline{x} = 8\,, \quad \overline{y} = 8\,, \quad \overline{w} = 0\,, \quad \overline{\lambda}_1 = 16\,, \quad \overline{\lambda}_2 = 0$$

- max $U = xyzw$

  s.t. $x + y + z + w = 4$

$$\Rightarrow \mathcal{L} = xyzw + \lambda(4 - x - y - z - w)$$

$$\Rightarrow \left.\begin{aligned} \mathcal{L}_\lambda &= 4 - x - y - z - w \overset{\text{set}}{=} 0 \\ \mathcal{L}_x &= yzw - \lambda \overset{\text{set}}{=} 0 \\ \mathcal{L}_y &= xzw - \lambda \overset{\text{set}}{=} 0 \\ \mathcal{L}_z &= xyw - \lambda \overset{\text{set}}{=} 0 \\ \mathcal{L}_w &= xyz - \lambda \overset{\text{set}}{=} 0 \end{aligned}\right\} \quad \textbf{[1st-order conditions]}$$

$$\Rightarrow \overline{x} = 1\,, \quad \overline{y} = 1\,, \quad \overline{z} = 1\,, \quad \overline{w} = 1\,, \quad \overline{\lambda} = 1$$

- **Determinantal test for a constrained extremum**

1. Suppose there are m constraints and n variables.
2. Verify the signs of $|\overline{H}_{m+1}|, \ |\overline{H}_{m+2}|, \ \cdots, \ |\overline{H}_n| \ (= |\overline{H}|)$
3. Positive definite $\begin{cases} m \text{ is even: } + \quad + \quad + \quad + \quad \cdots \\ m \text{ is odd: } - \quad - \quad - \quad - \quad \cdots \end{cases}$

   Negative definite $\begin{cases} m \text{ is even: } - \quad + \quad - \quad + \quad \cdots \\ m \text{ is odd: } + \quad - \quad + \quad - \quad \cdots \end{cases}$

- **2nd-order condition (the Bordered Hessian)**

  Case 1:

  $$\mathcal{L} = (x_1 x_2 + 2x_1) + \lambda(60 - 4x_1 - 2x_2)$$

  $\Rightarrow m = 1\ ,\quad n = 2$ and

  $$|\overline{H}| = \begin{vmatrix} 0 & -4 & -2 \\ -4 & 0 & 1 \\ -2 & 1 & 0 \end{vmatrix} \ ;$$

  $$\Rightarrow |\overline{H}_{1+1}| = |\overline{H}_2| = |\overline{H}| = 16 > 0$$

- Case 2:

$$\mathcal{L} = (x^2 + 2xy + yw^2) + \lambda_1(24 - 2x - y - w^2) + \lambda_2(8 - x - w)$$

$\Rightarrow m = 2 \ , \quad n = 3$ and

$$|\overline{H}| = \begin{vmatrix} 0 & 0 & -2 & -1 & -2w \\ 0 & 0 & -1 & 0 & -1 \\ -2 & -1 & 2 & 2 & 0 \\ -1 & 0 & 2 & 0 & 2w \\ -2w & -1 & 0 & 2w & 2y - 2\lambda_1 \end{vmatrix} ;$$

$$\Rightarrow |\overline{H}_{2+1}| = |\overline{H}_3| = |\overline{H}| = -22 < 0$$

- Case 3:

$$\mathcal{L} = xyzw + \lambda(4 - x - y - z - w)$$

$\Rightarrow m = 1 \,, \quad n = 4$ and

$$|\overline{H}| = \begin{vmatrix} 0 & -1 & -1 & -1 & -1 \\ -1 & 0 & zw & yw & yz \\ -1 & zw & 0 & xw & xz \\ -1 & yw & xw & 0 & xy \\ -1 & yz & xz & xy & 0 \end{vmatrix} \;;$$

$$\Rightarrow |\overline{H}_{1+1}| = |\overline{H}_2| = 2 \,, \quad |\overline{H}_3| = -3 \,, \quad |\overline{H}_4| = |\overline{H}| = 4$$

- max $U = U(x_1, x_2)$

  s.t. $p_1 x_1 + p_2 x_2 = m$

$$\Rightarrow \mathcal{L}(x_1, x_2, \lambda, p_1, p_2, m) = U(x_1,\ x_2) + \lambda(m - p_1 x_1 - p_2 x_2)$$

$$\Rightarrow \mathcal{L}_\lambda = m - p_1 x_1 - p_2 x_2 \overset{\text{set}}{=} 0$$

$$\left.\begin{aligned} \mathcal{L}_1 &= U_1 - \lambda p_1 \overset{\text{set}}{=} 0 \\ \mathcal{L}_2 &= U_2 - \lambda p_2 \overset{\text{set}}{=} 0 \end{aligned}\right\} \Rightarrow MRS_{12} = \frac{U_1}{U_2} = \frac{\lambda p_1}{\lambda p_2} = \frac{p_1}{p_2}$$

$$\Rightarrow |J| = \begin{vmatrix} 0 & -p_1 & -p_2 \\ -p_1 & U_{11} & U_{12} \\ -p_2 & U_{21} & U_{22} \end{vmatrix} \neq 0 \Rightarrow \begin{aligned} \overline{x}_1 &= \overline{x}_1(p_1, p_2, m) \\ \overline{x}_2 &= \overline{x}_2(p_1, p_2, m) \\ \overline{\lambda} &= \overline{\lambda}\,(p_1, p_2, m) \end{aligned}$$

- Define $\overline{\mathcal{L}}(p_1, p_2, m) \equiv \mathcal{L}(\overline{x}_1, \overline{x}_2, \overline{\lambda}, p_1, p_2, m)$

$$= U(\overline{x}_1, \overline{x}_2) + \overline{\lambda}(m - p_1\overline{x}_1 - p_2\overline{x}_2)$$

$$\Rightarrow \frac{\partial\overline{\mathcal{L}}}{\partial m} = U_1\frac{\partial\overline{x}_1}{\partial m} + U_2\frac{\partial\overline{x}_2}{\partial m} + \frac{\partial\overline{\lambda}}{\partial m}(m - p_1\overline{x}_1 - p_2\overline{x}_2) + \overline{\lambda}(1 - p_1\frac{\partial\overline{x}_1}{\partial m} - p_2\frac{\partial\overline{x}_2}{\partial m})$$

$$= (U_1 - \overline{\lambda}p_1)\frac{\partial\overline{x}_1}{\partial m} + (U_2 - \overline{\lambda}p_2)\frac{\partial\overline{x}_2}{\partial m} + (m - p_1\overline{x}_1 - p_2\overline{x}_2)\frac{\partial\overline{\lambda}}{\partial m} + \overline{\lambda}$$

$$= \overline{\lambda}$$

$\Rightarrow \overline{\lambda}$ measures the effect of a change in m on the optimal value of the objective function $\mathcal{L}$

$$\begin{aligned} 0d\lambda - p_1dx_1 - p_2dx_2 &= \overline{x}_1dp_1 + \overline{x}_2dp_2 - dm \\ -p_1d\lambda + U_{11}dx_1 + U_{12}dx_2 &= \overline{\lambda}dp_1 \\ -p_2d\lambda + U_{21}dx_1 + U_{22}dx_2 &= \overline{\lambda}dp_2 \end{aligned}$$

$$\Rightarrow \begin{bmatrix} 0 & -p_1 & -p_2 \\ -p_1 & U_{11} & U_{12} \\ -p_2 & U_{21} & U_{22} \end{bmatrix} \begin{bmatrix} d\lambda \\ dx_1 \\ dx_2 \end{bmatrix} = \begin{bmatrix} \overline{x}_1dp_1 + \overline{x}_2dp_2 - dm \\ \bar{\lambda}dp_1 \\ \bar{\lambda}dp_2 \end{bmatrix}$$

$$\Rightarrow dx_1 = \frac{1}{|J|} \begin{vmatrix} 0 & \overline{x}_1dp_1 + \overline{x}_2dp_2 - dm & -p_2 \\ -p_1 & \overline{\lambda}dp_1 & U_{12} \\ -p_2 & \overline{\lambda}dp_2 & U_{22} \end{vmatrix}$$

- **The Price Effect (Let $dm = dp_2 = 0$)**

$$\Rightarrow dx_1 = \frac{1}{|J|}\begin{vmatrix} 0 & \bar{x}_1 dp_1 & -p_2 \\ -p_1 & \bar{\lambda} dp_1 & U_{12} \\ -p_2 & 0 & U_{22} \end{vmatrix} = \frac{1}{|J|}\begin{vmatrix} 0 & \bar{x}_1 & -p_2 \\ -p_1 & \bar{\lambda} & U_{12} \\ -p_2 & 0 & U_{22} \end{vmatrix} dp_1$$

$$\Rightarrow \frac{\partial x_1}{\partial p_1} \equiv \left.\frac{dx_1}{dp_1}\right|_{dm=dp_2=0}$$

$$= \frac{1}{|J|}\left(-\bar{x}_1 \begin{vmatrix} -p_1 & U_{12} \\ -p_2 & U_{22} \end{vmatrix} + \bar{\lambda} \begin{vmatrix} 0 & -p_2 \\ -p_2 & U_{22} \end{vmatrix} \right)$$

- **The Income Effect (Let $dp_1 = dp_2 = 0$)**

$$\Rightarrow dx_1 = \frac{1}{|J|}\begin{vmatrix} 0 & -dm & -p_2 \\ -p_1 & 0 & U_{12} \\ -p_2 & 0 & U_{22} \end{vmatrix} = \frac{1}{|J|}\begin{vmatrix} 0 & -1 & -p_2 \\ -p_1 & 0 & U_{12} \\ -p_2 & 0 & U_{22} \end{vmatrix} dm$$

$$\Rightarrow \frac{\partial x_1}{\partial m} \equiv \left.\frac{dx_1}{dm}\right|_{dp_1 = dp_2 = 0}$$

$$= \frac{1}{|J|}\begin{vmatrix} -p_1 & U_{12} \\ -p_2 & U_{22} \end{vmatrix}$$

- **The Substitution Effect (Let $dU = 0$)**

$$U = U(x_1, x_2) \Rightarrow dU = U_1 dx_1 + U_2 dx_2 = 0$$

$$\Rightarrow \overline{\lambda}(p_1 dx_1 + p_2 dx_2) = 0$$

$$\Rightarrow \overline{x}_1 dp_1 + \overline{x}_2 dp_2 - dm = 0$$

$$\Rightarrow \left.\frac{\partial x_1}{\partial p_1}\right|_{U=\overline{U}} = \left.\frac{dx_1}{dp_1}\right|_{dU=0 \text{ and } dp_2=0}$$

$$= \frac{1}{|J|}\begin{vmatrix} 0 & 0 & -p_2 \\ -p_1 & \overline{\lambda} & U_{12} \\ -p_2 & 0 & U_{22} \end{vmatrix} = \frac{1}{|J|}\left(\overline{\lambda}\begin{vmatrix} 0 & -p_2 \\ -p_2 & U_{22} \end{vmatrix}\right) < 0$$

- **The Slutsky Equation**

$$\frac{\partial x_1}{\partial p_1} = \frac{1}{|J|}\left(-\overline{x}_1 \begin{vmatrix} -p_1 & U_{12} \\ -p_2 & U_{22} \end{vmatrix} + \overline{\lambda} \begin{vmatrix} 0 & -p_2 \\ -p_2 & U_{22} \end{vmatrix} \right)$$

$$= \frac{1}{|J|}\left(\overline{\lambda} \begin{vmatrix} 0 & -p_2 \\ -p_2 & U_{22} \end{vmatrix} \right) - \overline{x}_1 \left(\frac{1}{|J|} \begin{vmatrix} -p_1 & U_{12} \\ -p_2 & U_{22} \end{vmatrix} \right)$$

$$= \left. \frac{\partial x_1}{\partial p_1} \right|_{U=\overline{U}} - \overline{x}_1 \left(\frac{\partial x_1}{\partial m} \right)$$

- max $U = U(x_1, x_2)$

  s.t. $p_1 x_1 + p_2 x_2 = m$

$$\Rightarrow |\overline{H}| = \begin{vmatrix} 0 & -p_1 & -p_2 \\ -p_1 & U_{11} & U_{12} \\ -p_2 & U_{21} & U_{22} \end{vmatrix}$$

$$= -({p_1}^2 U_{22} - 2p_1 p_2 U_{12} + {p_2}^2 U_{11}) > 0$$

Let $U(x_1, x_2) = U_0 \quad \Rightarrow \quad U_1 dx_1 + U_2 dx_2 = dU_0 = 0$

$$\Rightarrow \frac{dx_2}{dx_1} = -\frac{U_1}{U_2} = -MRS_{12} < 0$$

$$\Rightarrow \frac{d^2x_2}{dx_1{}^2} \equiv \frac{d}{dx_1}\left(\frac{dx_2}{dx_1}\right)$$

$$= \frac{d}{dx_1}\left(-\frac{U_1}{U_2}\right) = -\frac{1}{U_2{}^2}\left(\frac{dU_1}{dx_1} \cdot U_2 - \frac{dU_2}{dx_1} \cdot U_1\right)$$

$$\because \frac{dU_1}{dx_1} = U_{11} + U_{12}\frac{dx_2}{dx_1} = U_{11} - \frac{U_{12}U_1}{U_2}$$

$$\frac{dU_2}{dx_1} = U_{21} + U_{22}\frac{dx_2}{dx_1} = U_{12} - \frac{U_{22}U_1}{U_2}$$

$$\Rightarrow \frac{d^2x_2}{dx_1{}^2} = -\frac{1}{U_2{}^3}(U_1{}^2U_{22} - 2U_1U_2U_{12} + U_2{}^2U_{11})$$

$$= -\frac{\lambda^2}{U_2{}^3}(p_1{}^2U_{22} - 2p_1p_2U_{12} + p_2{}^2U_{11}) > 0$$

- min $\mathcal{C} = wL + rK$

  s.t. $F(L, K) = Q_0$

$$\Rightarrow \mathcal{L} = wL + rK + \lambda[Q_0 - F(L, K)]$$

$$\Rightarrow \mathcal{L}_\lambda = Q_0 - F(L, K) \overset{\text{set}}{=} 0$$

$$\left.\begin{array}{l} \mathcal{L}_L = w - \lambda F_L \overset{\text{set}}{=} 0 \\ \mathcal{L}_K = r - \lambda F_K \overset{\text{set}}{=} 0 \end{array}\right\} \Rightarrow MRTS = \frac{F_L}{F_K} = \frac{w}{r}$$

Q: Write the bordered Hessian.

Q: Show all the iso-quant curves are negatively sloping and convex to the origin.

- **(Homogeneous Functions)**

  A function f defined on $\mathbb{R}^N$ is **homogeneous of degree r** if for every $t > 0$ we have

$$f(tx_1,\ tx_2, \cdots,\ tx_N) = t^r f(x_1, x_2, \cdots, x_N).$$

ex: $f(x, y, z) = \frac{x}{y} + \frac{2z}{3x}$

ex: $g(x, y, z) = \frac{x^2}{y} + \frac{yz}{x}$

ex: $h(x, y, z) = 2x^2 + 3xy - yz$

ex: $\mathcal{L}(x, y, z) = x^3 - 3xy + y^2 z$

- Suppose the production function $y = f(\mathbf{x})$, $\mathbf{x} \in \mathbb{R}^N_+$, is homogeneous of degree r, that is,

$$f(t\mathbf{x}) = t^r f(\mathbf{x})$$

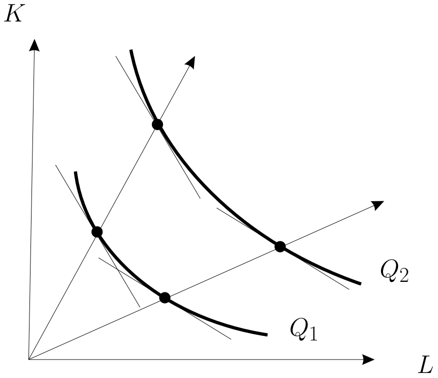
then this production function displays:

i. Increasing returns to scale if $r > 1$

ii. Constant returns to scale if $r = 1$

iii. Decreasing returns to scale if $r < 1$

- Suppose that $y = f(\mathbf{x}), \mathbf{x} \in \mathbb{R}_+^N$ is a homogeneous function. If $\mathbf{x}_1$ and $\mathbf{x}_2$ are any two points on the same level curve of the function f and we multiply each of these points by the same factor t to get points $t\mathbf{x}_1$ and $t\mathbf{x}_2$, respectively, then both of these points will also lie on a single-level curve.



- If f is homogeneous of degree r, then its first-order partial derivatives ($\partial f/\partial x_i$, $i = 1 \dots N$) are homogeneous of degree $r-1$.

Proof: Note that $f_i(t\mathbf{x}) \equiv \dfrac{\partial f(t\mathbf{x})}{\partial(tx_i)} \neq \dfrac{\partial f(t\mathbf{x})}{\partial x_i}$

$$f(tx_1, tx_2, \cdots, tx_N) = t^r f(x_1, x_2, \cdots, x_N)$$

$$\Rightarrow \frac{\partial}{\partial x_i}[f(tx_1, tx_2, \cdots, tx_N)] = \frac{\partial}{\partial x_i}[t^r f(x_1, x_2, \cdots, x_N)]$$

$$\Rightarrow \frac{\partial}{\partial(tx_i)}[f(tx_1, tx_2, \cdots, tx_N)]\,\frac{d(tx_i)}{dx_i} = t^r\,\frac{\partial}{\partial x_i}[f(x_1, x_2, \cdots, x_N)]$$

$$\Rightarrow f_i(tx_1, tx_2, \cdots, tx_N) = t^{r-1} f_i(x_1, x_2, \cdots, x_N)$$

ex: $f(x_1, x_2) = x_1^{1/3} x_2^{1/4} \quad \Rightarrow \quad f_1(x_1, x_2) = \frac{1}{3} x_1^{-2/3} x_2^{1/4}$

- If $Q = F(K, L)$ is a production function that is homogeneous of degree 1, then all its average and marginal products depend only on the capital-labor ratio.

Proof:

$$AP_L \equiv \frac{Q}{L} = \frac{1}{L}F(K, L) = F(\frac{K}{L}, \frac{L}{L}) = F(k, 1) = f(k)$$

$$AP_K \equiv \frac{Q}{K} = \frac{(Q/L)}{(K/L)} = f(k)/k$$

$$MP_L \equiv \frac{\partial Q}{\partial L} = \frac{\partial}{\partial L}[L \cdot f(k)] = f(k) + L \cdot f'(k) \cdot \frac{-K}{L^2}$$
$$= f(k) - kf'(k)$$

$$MP_K \equiv \frac{\partial Q}{\partial K} = \frac{\partial}{\partial K}[L \cdot f(k)] = L \cdot f'(k) \cdot \frac{1}{L} = f'(k)$$

- If $Q = F(K, L)$ is a production function which is homogeneous of degree r and has continuous first-order partial derivatives, then along any ray from the origin the slope of all isoquants, or the $MRTS$, is equal.

Proof:

Note the ratio K/L is constant along any ray from the origin.

$$MRTS(tK, tL) = \frac{MP_L(tK, tL)}{MP_K(tK, tL)} = \frac{t^{r-1} MP_L(K, L)}{t^{r-1} MP_K(K, L)}$$

$$= \frac{MP_L(K, L)}{MP_K(K, L)} = MRTS(K, L)$$

- **Euler's theorem**

  If $f(\mathbf{x}), \mathbf{x} \in \mathbb{R}^N_+$, is homogeneous of degree r, then the following condition holds:

$$f_1x_1 + f_2x_2 + \cdots + f_Nx_N = rf(x_1, x_2, \ldots, x_N)$$

Proof:

$$f(tx_1, tx_2, \cdots, tx_N) = t^r f(x_1, x_2, \cdots, x_N)$$

$$\Rightarrow \quad \frac{\partial}{\partial t}[f(tx_1, tx_2, \cdots, tx_N)] = \frac{\partial}{\partial t}[t^r f(x_1, x_2, \cdots, x_N)]$$

$$\Rightarrow \sum_{i=1}^{N}[\frac{\partial}{\partial(tx_i)} f(tx_1, tx_2, \cdots, tx_N)] \frac{\partial(tx_i)}{\partial t} = rt^{r-1} f(x_1, x_2, \cdots, x_N)$$

Since this condition holds for any $t > 0$, it also holds for $t = 1$

$$\Rightarrow \quad \sum_{i=1}^{N} f_i(x_1, x_2, \cdots, x_N) \cdot x_i = rf(x_1, x_2, \cdots, x_N)$$

- A function is **homothetic** if it is a monotonic transformation of some homogeneous function, that is,

$$f(x_1, x_2, \cdots, x_N) = h(g(x_1, x_2, \cdots, x_N))\text{ , where } h'(z) > 0$$

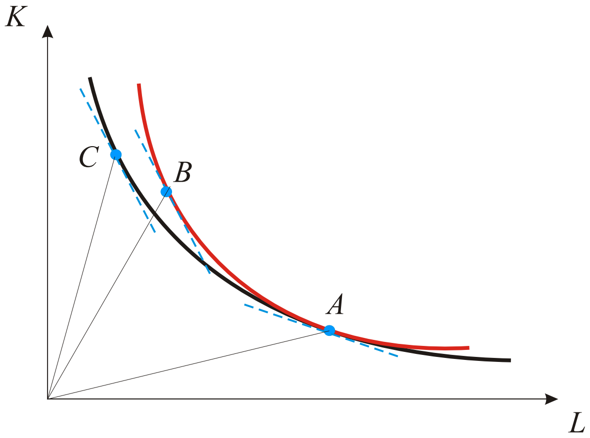
ex: $f(x_1, x_2) = 1 + {x_1}^{1/2}{x_2}^{1/2} \quad \Rightarrow \quad h(z) = 1 + z$

ex: $f(x_1, x_2) = ({x_1}^{2/3}{x_2}^{1/3})^r \ , \quad r > 0 \quad \Rightarrow \quad h(z) = z^r$

Thus, $\dfrac{f_1}{f_2} = \dfrac{h'(z) \cdot g_1}{h'(z) \cdot g_2} = \dfrac{g_1}{g_2}$

- The **elasticity of substitution** between inputs for a production function $Q = F(K, L)$ which has continuous marginal product functions is defined as

$$\sigma = \frac{d\ln(K/L)}{d\ln(w/r)}$$



$$\sigma \equiv \frac{\text{relative change in } (K/L)}{\text{relative change in } (w/r)}$$

$$= \frac{\dfrac{d(K/L)}{(K/L)}}{\dfrac{d(w/r)}{(w/r)}} = \frac{d\ln(K/L)}{d\ln(w/r)} = \frac{d\ln(K/L)}{d\ln(MRTS)}$$

ex: $F(K,L) = K^{2/3}L^{1/3}$

Integration

refer to textbook

Ch.14 Economic Dynamics and Integral Calculus

- Suppose that $\frac{d}{dx}F(x) = f(x)$. When the derivative f is known, we can determine the primitive function F.

$$\Rightarrow \int f(x)dx = F(x) + C$$

where $\int$ is the **integral sign**

$f(x)$ denotes the **integrand**

C is referred to as the **constant of integration**

- **Rules of indefinite integration**

Rule 1 (Power rule)

$$\int x^n dx = \frac{1}{n+1}x^{n+1} + C\ ,\quad n \neq -1$$

ex: $f(x) = x^3 \quad \Rightarrow \quad \int x^3 dx = \frac{1}{4}x^4 + C$

ex: $f(x) = 1 \quad \Rightarrow \quad \int 1 dx = x + C$

ex: $f(x) = \frac{1}{x^4} \quad \Rightarrow \quad \int x^{-4} dx = \frac{1}{(-3)}x^{-3} + C$

ex: $f(x) = \sqrt{x^3} \quad \Rightarrow \quad \int x^{3/2} dx = \frac{2}{5}x^{5/2} + C$

Rule 2 (Exponential rule)

$$\int e^x dx = e^x + C$$

and $$\int f'(x)e^{f(x)}dx = e^{f(x)} + C$$

ex: $f(x) = 2e^{2x} \ \Rightarrow \ \int 2e^{2x}dx = e^{2x} + C$

ex: $f(x) = (2x)\exp(x^2) \ \Rightarrow \ \int (2x)\exp(x^2)dx = \exp(x^2) + C$

Rule 3 (Logarithmic rule)

$$\int \frac{1}{x} dx = \ln x + C \ , \quad x > 0$$

and $$\int \frac{g'(x)}{g(x)} dx = \ln g(x) + C \ , \quad g(x) > 0$$

ex: $f(x) = \dfrac{2}{x} \quad \Rightarrow \quad \displaystyle\int \frac{2}{x} dx = 2 \ln x + C \ , \quad x > 0$

ex: $f(x) = \dfrac{14x}{7x^2+5} \quad \Rightarrow \quad \displaystyle\int \frac{14x}{7x^2+5} dx = \ln(7x^2+5) + C$

ex: $f(x) = \dfrac{x}{x^2-1}$

$$\Rightarrow \int \frac{x}{x^2-1} dx = \begin{cases} \frac{1}{2} \ln(x^2-1) + C, & x > 1 \text{ or } x < -1 \\ \frac{1}{2} \ln(1-x^2) + C, & -1 < x < 1 \end{cases}$$

Rule 4 (integral of a sum)

$$\int [f(x) + g(x)]dx = \int f(x)dx + \int g(x)dx$$

Rule 5 (integral of a constant multiple)

$$\int kf(x)dx = k \int f(x)dx$$

ex:

$$\begin{aligned} \int (3x^2 + 8x^5)dx &= 3 \int x^2 dx + 8 \int x^5 dx \\ &= 3(\frac{1}{3}x^3 + C_1) + 8(\frac{1}{6}x^6 + C_2) \\ &= x^3 + \frac{4}{3}x^6 + C \end{aligned}$$

Rule 6 (the substitution rule)

$$\int [f(u) \cdot (\frac{du}{dx})] dx = F(u) + C$$

Proof

$$\frac{d}{dx}F(u) = [\frac{d}{du}F(u)] \cdot (\frac{du}{dx}) = f(u) \cdot (\frac{du}{dx})$$

ex: $\int 6x^2(x^3+2)^{99}dx \quad \Rightarrow$ Let $u = x^3 + 2$, then $\frac{du}{dx} = 3x^2$

$$= \int 2(3x^2)(x^3+2)^{99}dx = 2\int u^{99}(\frac{du}{dx})dx$$

$$= \frac{2}{100}u^{100} + C = \frac{1}{50}(x^3+2)^{100} + C$$

Rule 7 (Integration by parts)

$$\int v du = uv - \int u dv$$

Proof

$$d(uv) = v du + u dv$$

$$\Rightarrow \int d(uv) = \int v du + \int u dv$$

$$\Rightarrow uv = \int v du + \int u dv$$

ex: $\int x(x+1)^{1/2}dx \quad \Rightarrow$ Let $v = x$ and $du = (x+1)^{1/2}dx$,

then $dv = dx$ and $u = \frac{2}{3}(x+1)^{3/2}$

$$= x\left[\frac{2}{3}(x+1)^{3/2}\right] - \int \frac{2}{3}(x+1)^{3/2}dx$$

$$= \tfrac{2}{3}x(x+1)^{3/2} - \tfrac{4}{15}(x+1)^{5/2} + C$$

ex: $\int \ln x dx \quad \Rightarrow$ Let $v = \ln x$ and $du = dx$, then $dv = \frac{1}{x}dx$ and $u = x$

$$= x \ln x - \int x(\frac{1}{x}dx) = x \ln x - x + C$$

ex: $\int xe^x dx \quad \Rightarrow$ Let $v = x$ and $du = e^x dx$, then $dv = dx$ and $u = e^x$

$$= xe^x - \int e^x dx = xe^x - e^x + C$$

Differential Equations

refer to textbook

Ch.15 Continuous Time: First-Order Differential Equations

Ch.16 Higher-Order Differential Equations

- **First-Order Linear Differential Equations**

$$\frac{dy}{dt} + u(t)y = w(t)$$

or

$$\dot{y} + u(t)y = w(t)$$

Note that $(\ dy/dt\ )$ $\rightarrow$ 1st-order

$(\ d^2y/dt^2\ )$ $\rightarrow$ 2nd-order

$(\ dy/dt\ )^1$ $\rightarrow$ degree 1

$(\ dy/dt\ )^r$ $\rightarrow$ degree r

Case 1 (Homogeneous with Constant Coefficients)

ex: $\dfrac{dy}{dt} + 4y = 0$

$\Rightarrow \dfrac{dy}{dt} = -4y$ or $\dfrac{1}{y}dy = -4dt$

$\Rightarrow \displaystyle\int \frac{1}{y}dy = \int (-4)dt$

$\Rightarrow \ln|y| = -4t + C$ or $|y| = e^{-4t+C}$

$\Rightarrow y(t) = \pm e^{-4t} \cdot e^{C} = \pm Ae^{-4t}$ [general solution]

$= y(0)e^{-4t}$ [definite solution]

Case 2 (Nonhomogeneous with Constant Coefficients)

ex: $\frac{dy}{dt} + 2y = 6 \quad \Rightarrow$

(reduced eq.) $\frac{dy}{dt} + 2y = 0$

$\Longrightarrow \; y_c = Ae^{-2t}$ complementary function

(complete eq.) $\frac{dy}{dt} + 2y = 6$

$\Longrightarrow \; y_p = 3$ (try $y = k$) particular integral

$\Rightarrow \; y(t) = y_c + y_p = Ae^{-2t} + 3$ [general solution]

$= [y(0) - 3]e^{-2t} + 3$ [definite solution]

proof:

$$\frac{dy}{dt} + ay = b \quad \Rightarrow \quad y_p$$

$$\frac{dy}{dt} + ay = 0 \quad \Rightarrow \quad y_c$$

Let $y = y_p + y_c$, then

$$\frac{dy}{dt} = \frac{d}{dt}(y_p + y_c) = \frac{dy_p}{dt} + \frac{dy_c}{dt}$$

$$ay = a(y_p + y_c) = ay_p + ay_c$$

$$\Rightarrow \frac{dy}{dt} + ay = (\frac{dy_p}{dt} + ay_p) + (\frac{dy_c}{dt} + ay_c) = b$$

ex: $\frac{dy}{dt} = 2$

Way 1

$$\int dy = \int 2dt \;\Rightarrow\; y(t) = 2t + C = y(0) + 2t$$

Way 2

$$\frac{dy}{dt} = 0 \quad \Longrightarrow \quad y_c = A$$

$$\frac{dy}{dt} = 2 \quad \Longrightarrow \quad y_p = 2t$$

try $y = kt$

$$\Rightarrow \; y(t) = y_c + y_p = A + 2t = y(0) + 2t$$

Case 3 (Homogeneous with Variable Coefficients)

ex: $\dfrac{dy}{dt} + (3t^2)y = 0$

$$\Rightarrow \int \frac{1}{y} dy = \int (-3t^2) dt$$

$$\Rightarrow \ln|y| = -t^3 + C$$

$$\Rightarrow y(t) = \pm A e^{-t^3} = y(0) e^{-t^3}$$

Case 4 (Nonhomogeneous with Variable Coefficients)

- Exact Differential Equations

  We say that

$$M\ dy + N\ dt = 0$$

is **exact** if and only if there exists a function $F(y, t)$ such that

$M = \dfrac{\partial F}{\partial y}$ and $N = \dfrac{\partial F}{\partial t}$, (or $\dfrac{\partial M}{\partial t} = \dfrac{\partial N}{\partial y}$ is met)

$$\Rightarrow\ dF(y,t) = \frac{\partial F}{\partial y}dy + \frac{\partial F}{\partial t}dt = 0$$

Step 1 $F(y,t) = \int M dy + \psi(t)$

Step 2 $\frac{\partial}{\partial t}[\int M dy + \psi(t)] = N$

Step 3 Solve for $\psi(t)$

Step 4 Replace $\psi(t)$ into $F(y,t)$ and then $F(y,t) = C$ will be the solution.

ex: $(2yt)dy + y^2 dt = 0$

$$\Rightarrow \frac{\partial}{\partial t}(2yt) = 2y = \frac{\partial}{\partial y}(y^2) \qquad \text{Exact !}$$

Step 1 $F(y,t) = \int (2yt)dy + \psi(t) = ty^2 + C_1 + \psi(t)$

Step 2 $\frac{\partial}{\partial t}[ty^2 + C_1 + \psi(t)] = y^2 + \psi'(t) = y^2 \;\Rightarrow\; \psi'(t) = 0$

Step 3 $\psi(t) = C_2$

Step 4 $F(y,t) = ty^2 + C_1 + C_2 = C_3$

$$\Rightarrow\; ty^2 = C \quad \text{or} \quad y(t) = \pm\sqrt{\frac{C}{t}}$$

ex: $(t+2y)dy + (y+3t^2)dt = 0$

$$\Rightarrow \frac{\partial}{\partial t}(t+2y) = 1 = \frac{\partial}{\partial y}(y+3t^2) \qquad \text{Exact !}$$

Step 1 $F(y,t) = \int (t+2y)dy + \psi(t) = ty + y^2 + C_1 + \psi(t)$

Step 2 $\frac{\partial}{\partial t}[ty + y^2 + C_1 + \psi(t)] = y + \psi'(t) = y + 3t^2$

$$\Rightarrow \psi'(t) = 3t^2$$

Step 3 $\psi(t) = t^3 + C_2$

Step 4 $F(y,t) = ty + y^2 + C_1 + t^3 + C_2 = C_3$

$$\Rightarrow y^2 + ty + (t^3 - C) = 0$$

$$\Rightarrow y(t) = \frac{-t \pm \sqrt{t^2 - 4(t^3 - C)}}{2}$$

- What if $\frac{\partial}{\partial t}M \neq \frac{\partial}{\partial y}N$?

ex: $(2t)dy + ydt = 0$

$$\Rightarrow \frac{\partial}{\partial t}(2t) = 2 \neq 1 = \frac{\partial}{\partial y}y$$

ex: $2(t^3 + 1)dy + (3yt^2)dt = 0$

$$\Rightarrow \frac{\partial}{\partial t}(2t^3 + 2) = 6t^2 \neq 3t^2 = \frac{\partial}{\partial y}(3yt^2)$$

ex: $(4y^3t)dy + (2y^4 + 3t)dt = 0$

$$\Rightarrow \frac{\partial}{\partial t}(4y^3t) = 4y^3 \neq 8y^3 = \frac{\partial}{\partial y}(2y^4 + 3t)$$

$\Rightarrow$ Look for the possible **Integrating Factors** !

ex: $(2ty)dy + y^2 dt = 0$

$$\Rightarrow \frac{\partial}{\partial t}(2ty) = 2y = \frac{\partial}{\partial y}(y^2)$$

ex: $2(t^3+1)y dy + (3y^2 t^2)dt = 0$

$$\Rightarrow \frac{\partial}{\partial t}[2(t^3+1)y] = 6t^2 y = \frac{\partial}{\partial y}(3y^2 t^2)$$

ex: $(4y^3 t^2)dy + (2y^4 t + 3t^2)dt = 0$

$$\Rightarrow \frac{\partial}{\partial t}(4y^3 t^2) = 8y^3 t = \frac{\partial}{\partial y}(2y^4 t + 3t^2)$$

- Integrating Factors

$$\frac{dy}{dt} + u(t)y = w(t) \;\Rightarrow\; dy + [u(t)y - w(t)]dt = 0$$

$$\Rightarrow\; I(t)dy + I(t)[u(t)y - w(t)]dt = 0$$

$$\Rightarrow\; \frac{\partial}{\partial t}I(t) = \frac{\partial}{\partial y}\big(I(t)[u(t)y - w(t)]\big) = I(t)u(t)$$

$$\Rightarrow\; \int \frac{1}{I}dI = \int u(t)dt = \ln|I|$$

$$\Rightarrow\; I(t) = \exp[\int u(t)dt]$$

ex: $2tdy + ydt = 0$

$$\Rightarrow \frac{dy}{dt} + (\frac{1}{2t})y = 0 \Rightarrow u(t) = \frac{1}{2t}$$

$$\Rightarrow \text{I.F.} = \exp[\int \frac{1}{2t}dt] = e^{\frac{1}{2}\ln t} = t^{\frac{1}{2}}$$

Check:

$$dy + (\frac{1}{2t})ydt = 0$$

$$\Rightarrow t^{\frac{1}{2}}dy + (\frac{1}{2}t^{-\frac{1}{2}})ydt = 0$$

$$\Rightarrow \frac{\partial}{\partial t}(t^{\frac{1}{2}}) = \frac{1}{2}t^{-\frac{1}{2}} = \frac{\partial}{\partial y}[(\frac{1}{2}t^{-\frac{1}{2}})y]$$

ex: $2(t^3+1)dy + 3yt^2dt = 0$

$$\Rightarrow \frac{dy}{dt} + \frac{3t^2}{2(t^3+1)}y = 0 \quad \Rightarrow \quad u(t) = \frac{3t^2}{2(t^3+1)}$$

$$\Rightarrow \text{ I.F.} = \exp[\int \frac{3t^2}{2(t^3+1)}dt] = e^{\frac{1}{2}\ln(t^3+1)} = (t^3+1)^{\frac{1}{2}}$$

Check:

$$dy + \frac{3t^2}{2(t^3+1)}ydt = 0$$

$$\Rightarrow (t^3+1)^{\frac{1}{2}}dy + \frac{3}{2}t^2(t^3+1)^{-\frac{1}{2}}ydt = 0$$

$$\Rightarrow \frac{\partial}{\partial t}[(t^3+1)^{\frac{1}{2}}] = \frac{1}{2}(t^3+1)^{-\frac{1}{2}}(3t^2) = \frac{\partial}{\partial y}[\frac{3}{2}t^2(t^3+1)^{-\frac{1}{2}}]$$

- Bernoulli Equation

$$\frac{dy}{dt} + R(t)y = F(t)y^m \ , \quad m \neq 0, \ 1.$$

$$\Rightarrow \ y^{-m} \cdot \frac{dy}{dt} + R(t)y^{1-m} = F(t)$$

Let $z = y^{1-m}$, so that

$$\frac{dz}{dt} = (\frac{dz}{dy})(\frac{dy}{dt}) = (1-m)y^{-m}(\frac{dy}{dt})$$

$$\Rightarrow \ \frac{1}{1-m} \cdot \frac{dz}{dt} + R(t)z = F(t)$$

or $$\frac{dz}{dt} + (1-m)R(t)z = (1-m)F(t)$$

ex: $\frac{dy}{dt} + (\frac{1}{t})y = y^3 \quad \Rightarrow \quad y^{-3}\frac{dy}{dt} + (\frac{1}{t})y^{-2} = 1$

$\Rightarrow$ Let $z = y^{-2}$, so that $\frac{dz}{dt} = (\frac{dz}{dy})(\frac{dy}{dt}) = (-2)y^{-3}(\frac{dy}{dt})$

$\Rightarrow \frac{1}{(-2)}\frac{dz}{dt} + (\frac{1}{t})z = 1$ or $\frac{dz}{dt} + (\frac{-2}{t})z = -2$

$\Rightarrow$ I.F. $= \exp[\int(\frac{-2}{t})dt] = e^{-2\ln t} = \frac{1}{t^2}$

$\Rightarrow (\frac{1}{t^2})dz + [(\frac{-2}{t^3})z + 2(\frac{1}{t^2})]dt = 0$

Check:

$$\frac{\partial}{\partial t}\left[\frac{1}{t^2}\right] = (-2)t^{-3} = \frac{\partial}{\partial z}[(\frac{-2}{t^3})z + 2(\frac{1}{t^2})]$$

$$(\frac{1}{t^2})dz + \left[(\frac{-2}{t^3})z + 2(\frac{1}{t^2})\right]dt = 0$$

Step 1 $F(z,t) = \int (\frac{1}{t^2})dz + \psi(t) = t^{-2}z + \psi(t)$

Step 2 $\frac{\partial}{\partial t}[t^{-2}z + \psi(t)] = (-2)t^{-3}z + \psi'(t) = (\frac{-2}{t^3})z + 2(\frac{1}{t^2})$

$\Rightarrow \psi'(t) = 2t^{-2}$

Step 3 $\psi(t) = (-2)t^{-1}$

Step 4 $F(z,t) = t^{-2}z + (-2)t^{-1} = C$

$$\Rightarrow \; z = 2t + Ct^2 = y^{-2}$$

$$\Rightarrow \; y(t) = \pm\sqrt{\frac{1}{2t + Ct^2}}$$

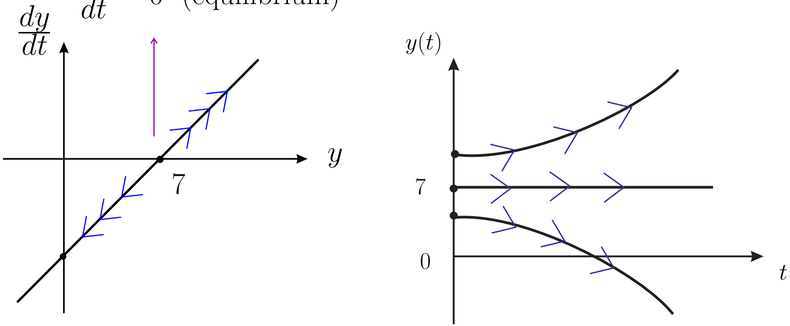
- Phase Diagram $\frac{dy}{dt} = f(y)$

ex: $\frac{dy}{dt} = y - 7 \quad \Rightarrow \quad \frac{dy}{dt} - y = -7$

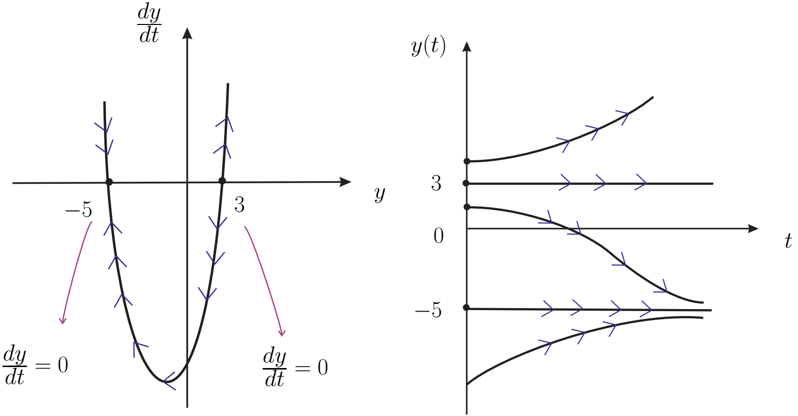
$\Rightarrow\ y_c = Ae^t$ and $y_p = 7$

$\Rightarrow\ y(t) = Ae^t + 7 = [y(0) - 7]e^t + 7$

$\frac{dy}{dt} = 0$ (equilibrium)



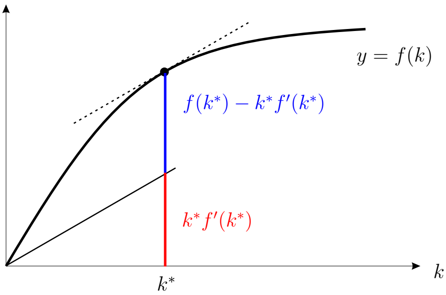
ex: $\frac{dy}{dt} = (y+1)^2 - 16$



Solow Growth Model

- $Y = F(K, L) \quad \xrightarrow{CRTS} \quad Y = L \cdot F(\frac{K}{L}, \frac{L}{L}) = L \cdot f(k)$

  or $\quad y = f(k)$



$$MP_L \equiv \frac{\partial Y}{\partial L} = f(k) - kf'(k)$$

$$MP_K \equiv \frac{\partial Y}{\partial K} = f'(k)$$

$$F_{KK} \equiv \frac{\partial}{\partial K} f'(k) = \frac{f''(k)}{L}$$

Solow Growth Model

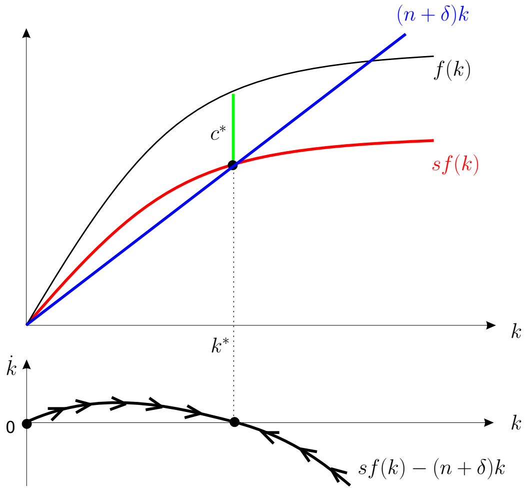
- $I = \frac{dK}{dt} + \delta K = \dot{K} + \delta K \qquad (0 < \delta < 1)$
- $S = sY \qquad (0 < s < 1) \qquad I \overset{\text{set}}{=} S$
- $\gamma_L \equiv \frac{\dot{L}}{L} = n$

$\Rightarrow \; sY = \dot{K} + \delta K = (\gamma_K + \delta)K \qquad$ (Note that $\gamma_K = \gamma_k + \gamma_L$)

$\Rightarrow \qquad sy = (\gamma_k + n + \delta)k = \dot{k} + (n + \delta)k$

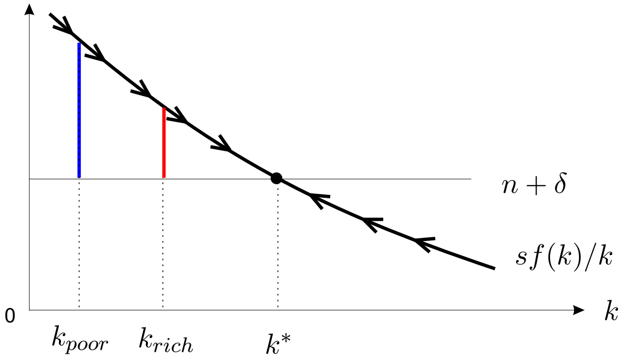
or $\quad \dot{k} = sf(k) - (n + \delta)k \qquad$ **[Solow equation]**

Solow Growth Model



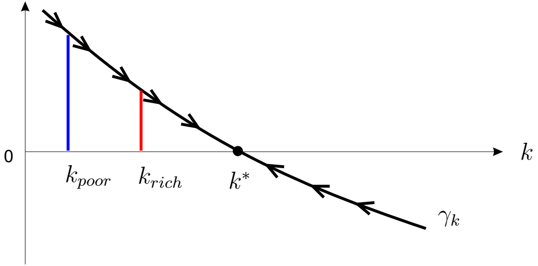
Solow Growth Model

- $\gamma_k \equiv \dot{k}/k = sf(k)/k - (n + \delta)$



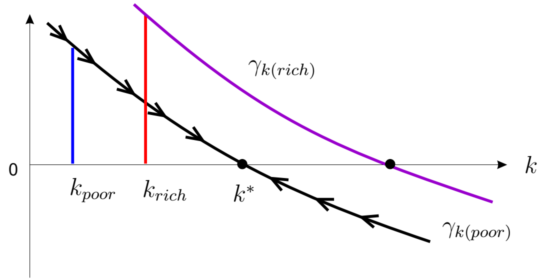
Solow Growth Model

- $\gamma_k \equiv \dot{k}/k = sf(k)/k - (n+\delta)$



- Hypothesis: **Poor Economies tend to grow faster per capita than rich ones**.

Solow Growth Model



- saving rate
- depreciation rate
- production function
- population growth rate

Solow Growth Model

ex: $\dot{k} = sk^{0.7} - (n+\delta)k$

$\Rightarrow \dfrac{dk}{dt} + (n+\delta)k = sk^{0.7}$ or $k^{-0.7}(\dfrac{dk}{dt}) + (n+\delta)k^{0.3} = s$

Let $z = k^{0.3}$ so that $\dfrac{dz}{dt} = 0.3k^{-0.7}(\dfrac{dk}{dt})$

hence $\dfrac{dz}{dt} + 0.3(n+\delta)z = 0.3s$

$\Rightarrow z(t) = \dfrac{s}{n+\delta} + [z(0) - \dfrac{s}{n+\delta}]\, e^{-0.3(n+\delta)t}$ or

$$k(t) = \left\{\frac{s}{n+\delta} + [k(0)^{0.3} - \frac{s}{n+\delta}]\, e^{-0.3(n+\delta)t}\right\}^{\frac{1}{0.3}}$$

Solow Growth Model

ex: Maximize $c^* = f(k^*) - sf(k^*) = (1-s)f(k^*)$

$\Rightarrow$ Since $sf(k^*) - (n+\delta)k^* = 0$ at equilibrium (**WHY?**)

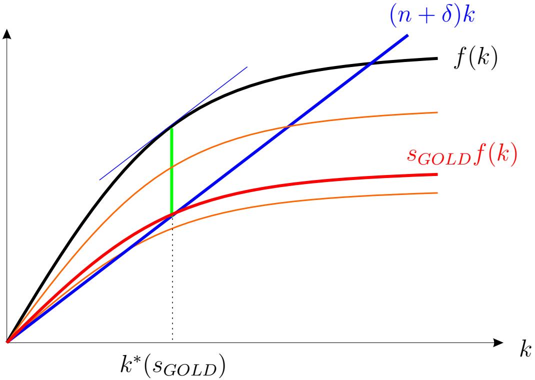
therefore, $k^* = k^*(s)$ and $\dfrac{dk^*}{ds} = -\dfrac{f(k^*)}{sf'(k^*) - (n+\delta)}$

$$\Rightarrow \frac{dc^*}{ds} = -f(k^*) + (1-s)f'(k^*) \cdot (\frac{d}{ds}k^*)$$

$$= -f(k^*) + (1-s)f'(k^*) \cdot \left(-\frac{f(k^*)}{sf'(k^*) - (n+\delta)}\right)$$

$$= \left(-\frac{f(k^*)}{sf'(k^*) - (n+\delta)}\right) \cdot [f'(k^*) - (n+\delta)]$$

Solow Growth Model



Nth-Order Linear Differential Equations

$$\frac{d^n y}{dt^n} + a_1 \frac{d^{n-1} y}{dt^{n-1}} + \cdots + a_{n-1} \frac{dy}{dt} + a_n y = b$$

or $$y^{(n)}(t) + a_1 y^{(n-1)}(t) + \cdots + a_{n-1} y'(t) + a_n y = b$$

1. Look for the **particular integral**: y_p

ex: $y''(t) + y'(t) - 2y(t) = -10$ try $\overrightarrow{y_p} = k$ $y_p = 5$ **O**

ex: $y''(t) + y'(t) = -10$ try $\overrightarrow{y_p} = k$ $0 = -10$ **X**

try $\overrightarrow{y_p} = kt$ $y_p = -10t$ **O**

ex: $y''(t) = -10$

2. Solve the **complementary function**: y_c

$$y''(t) + a_1 y'(t) + a_2 y(t) = 0$$

- Let $y_c = Ae^{rt}$, so that $y'(t) = rAe^{rt}$ and $y''(t) = r^2 Ae^{rt}$

$\Rightarrow$ $Ae^{rt}(r^2 + a_1 r + a_2) = 0$, we call $r^2 + a_1 r + a_2 = 0$ as a characteristic (or auxiliary) equation. (**Can** $A = 0$ **happen?)**

$$\Rightarrow r_1,\ r_2 = \frac{-a_1 \pm \sqrt{{a_1}^2 - 4a_2}}{2} \quad \Rightarrow \quad y_1 = A_1 e^{r_1 t}, \quad y_2 = A_2 e^{r_2 t}$$

$$\Rightarrow y_c = y_1 + y_2 = A_1 e^{r_1 t} + A_2 e^{r_2 t}$$

(Why not just pick any one of them?)

- Case 1. Two distinct real roots ($a_1^2 > 4a_2$)

ex: $y''(t) + y'(t) - 2y(t) = -10$

$\Rightarrow\ r^2 + r - 2 = (r+2)(r-1) = 0 \quad \Rightarrow \quad r_1 = 1,\ r_2 = -2$

$\Rightarrow\ y_c = A_1 e^{1t} + A_2 e^{-2t}$ and $y(t) = y_c + y_p = A_1 e^{1t} + A_2 e^{-2t} + 5$

If we let $y(0) = 12$ and $y'(0) = -2,$ then

$A_1 + A_2 + 5 = 12$ and $A_1 + (-2)A_2 = -2$

$\Rightarrow\ A_1 = 4,\ A_2 = 3,$ and $y(t) = 4e^{1t} + 3e^{-2t} + 5$

- Case 2. Two repeated real roots ($a_1^2 = 4a_2 \quad \Rightarrow \quad r = -\frac{a_1}{2}$)

ex: $y''(t) + 6y'(t) + 9y(t) = 27$

$\Rightarrow \; r^2 + 6r + 9 = (r+3)^2 = 0 \qquad \Rightarrow \qquad r_1 = r_2 = -3$

$\Rightarrow \; y_c = A_1 e^{-3t} + A_2 e^{-3t} = A_3 e^{-3t}$

(**Only one constant can be identified!**)

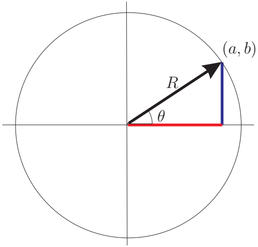
If we let $y_c = A_4 t e^{rt}$ (**Can it be another solution?**)

then $y(t) = (A_3 + A_4 t)e^{-3t} + 3$

(**Solve the definite solution given** $y(0) = 5$ **and** $y'(0) = -5$)

Trigonometric Functions and Complex Numbers

$$\begin{aligned} Z = a + bi &= \sqrt{a^2+b^2}\left(\frac{a}{\sqrt{a^2+b^2}} + \frac{b}{\sqrt{a^2+b^2}}\, i\right) \\ &= R(\cos\theta + i\sin\theta) \end{aligned}$$



- $\sin^2\theta + \cos^2\theta = 1$
- $\sin(\theta_1 \pm \theta_2) = \sin\theta_1\cos\theta_2 \pm \cos\theta_1\sin\theta_2$

  $\cos(\theta_1 \pm \theta_2) = \cos\theta_1\cos\theta_2 \mp \sin\theta_1\sin\theta_2$
- $Z_1 Z_2 = R_1 R_2\left(\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2)\right)$
- $Z^n = R^n\left(\cos n\theta + i\sin n\theta\right)$
- $\frac{d}{d\theta}\sin\theta = \cos\theta, \qquad \frac{d}{d\theta}\cos\theta = -\sin\theta$

Trigonometric Functions and Complex Numbers

$f(\theta) = \sin\theta$	$f(0) = 0$	$g(\theta) = \cos\theta$	$g(0) = 1$
$f'(\theta) = \cos\theta$	$f'(0) = 1$	$g'(\theta) = -\sin\theta$	$g'(0) = 0$
$f''(\theta) = -\sin\theta$	$f''(0) = 0$	$g''(\theta) = -\cos\theta$	$g''(0) = -1$
$f'''(\theta) = -\cos\theta$	$f'''(0) = -1$	$g'''(\theta) = \sin\theta$	$g'''(0) = 0$
$f^{(4)}(\theta) = \sin\theta$	$f^{(4)}(0) = 0$	$g^{(4)}(\theta) = \cos\theta$	$g^{(4)}(0) = 1$
$\vdots$	$\vdots$	$\vdots$	$\vdots$

$$\sin\theta = 0 + \frac{1}{1!}\theta + \frac{0}{2!}\theta^2 + \frac{-1}{3!}\theta^3 + \frac{0}{4!}\theta^4 + \frac{1}{5!}\theta^5 + \cdots + \frac{f^{(n)}(p)}{(n+1)!}\theta^{n+1}$$

0 as $n \to \infty$

$$= \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!} + \cdots$$

$$\cos\theta = 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \cdots$$

Trigonometric Functions and Complex Numbers

$$
\begin{aligned}
e^x &= 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \cdots \\
e^{(i\theta)} &= 1 + \frac{(i\theta)}{1!} + \frac{(i\theta)^2}{2!} + \frac{(i\theta)^3}{3!} + \frac{(i\theta)^4}{4!} + \frac{(i\theta)^5}{5!} + \cdots \\
&= 1 + i\theta - \frac{\theta^2}{2!} - \frac{i\theta^3}{3!} + \frac{\theta^4}{4!} + \frac{i\theta^5}{5!} \cdots \\
&= \left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} \cdots\right) + i\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!} \cdots\right) \\
&= \cos\theta + i\sin\theta \\
e^{(-i\theta)} &= \cos\theta - i\sin\theta
\end{aligned}
$$

$$
Z = a \pm bi = R(\cos\theta \pm i\sin\theta) = Re^{\pm i\theta}
$$

cartesian form   polar form   exponential form

- Case 3. Two (conjugate) complex roots ($a_1^2 < 4a_2$)

$$r_1,\ r_2 = \frac{-a_1 \pm \sqrt{4a_2 - {a_1}^2}\ i}{2} = \alpha \pm \beta i$$

$$\begin{aligned} y_c &= A_1 e^{(\alpha+\beta i)t} + A_2 e^{(\alpha-\beta i)t} \\ &= e^{\alpha t}\left(A_1 e^{i\beta t} + A_2 e^{-i\beta t}\right) \\ &= e^{\alpha t}\left[A_1(\cos\beta t + i\sin\beta t) + A_2(\cos\beta t - i\sin\beta t)\right] \\ &= e^{\alpha t}\left[(A_1 + A_2)\cos\beta t + (A_1 - A_2)i\sin\beta t\right] \\ &= e^{\alpha t}\left[A_5\cos\beta t + A_6\sin\beta t\right] \end{aligned}$$

- Case 3. Two (conjugate) complex roots ($a_1^2 < 4a_2$)

ex: $y''(t) + 2y'(t) + 17y(t) = 34, \quad y(0) = 3, \quad y'(0) = 11$

$\Rightarrow r^2 + 2r + 17 = 0 \quad \Rightarrow \quad r = -1 \pm 4i$

$\Rightarrow y(t) = e^{-t}(A_5 \cos 4t + A_6 \sin 4t) + 2$

and $y'(t) = -e^{-t}(A_5 \cos 4t + A_6 \sin 4t) + 4e^{-t}(-A_5 \sin 4t + A_6 \cos 4t)$

$\because y(0) = A_5 + 2 = 3$ and $y'(0) = -A_5 + 4A_6 = 11$

$$\begin{aligned} y(t) &= e^{-t}(\cos 4t + 3 \sin 4t) + 2 \\ &= \sqrt{10}e^{-t}\left(\frac{1}{\sqrt{10}} \cos 4t + \frac{3}{\sqrt{10}} \sin 4t\right) + 2 \\ &= \sqrt{10}e^{-t} \sin(4t + \phi) + 2 \end{aligned}$$

The Dynamic Stability at Equilibrium

- Case 1. Two distinct real roots

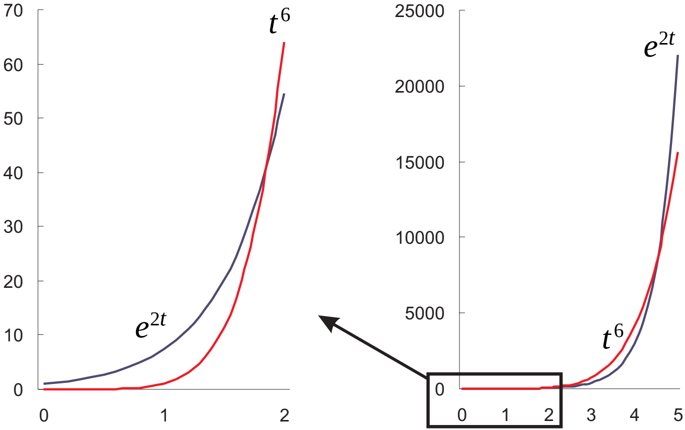
$$y_c = A_1 e^{r_1 t} + A_2 e^{r_2 t}$$

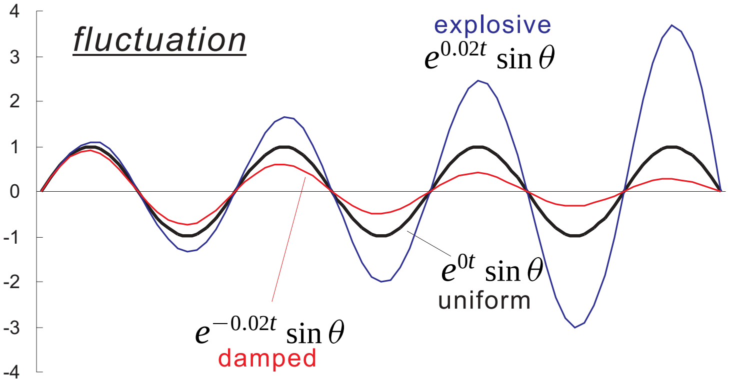
- Case 2. Two repeated real roots

$$y_c = (A_3 + A_4 t)e^{rt}$$

- Case 3. Two (conjugate) complex roots

$$y_c = e^{\alpha t}(A_5 \cos \beta t + A_6 \sin \beta t)$$


70
60
50
40
30
20
10
0
0
1
2
t^6
e^{2t}
25000
20000
15000
10000
5000
0
0
1
2
3
4
5
e^{2t}
t^6



4
3
2
1
0
-1
-2
-3
-4
fluctuation
explosive
$e^{0.02t} \sin\theta$
$e^{0t} \sin\theta$
uniform
$e^{-0.02t} \sin\theta$
damped

Differential Equations with a Variable Term

ex: $y'' + 5y' + 3y = 6t^2 - t - 1 \quad \Rightarrow \quad y_p$**?**

$$\begin{array}{rcrcrcr l}
y & = & at^2 & + & bt & + & c & \ldots \times \mathbf{3} \\
y' & = & & & 2at & + & b & \ldots \times \mathbf{5} \\
y'' & = & & & & + & 2a & \ldots \times \mathbf{1}
\end{array}$$

$$6t^2 - t - 1 \quad = \quad 3at^2 + (10a + 3b)t + (2a + 5b + 3c)$$

$\Rightarrow \ a = 2, \quad b = -7, \quad c = 10$

$\Rightarrow \ y_p = 2t^2 - 7t + 10$ **O**

Differential Equations with a Variable Term

ex: $y'' + 5y' = 6t^2 - t - 1 \quad \Rightarrow \quad y_p$**?**

$$
\begin{array}{rcrcrcrl}
y & = & at^2 & + & bt & + & c & \ldots \times \mathbf{0} \\
y' & = & & & 2at & + & b & \ldots \times \mathbf{5} \\
y'' & = & & & & + & 2a & \ldots \times \mathbf{1} \\
\hline
6t^2 - t - 1 & = & \multicolumn{5}{l}{10at + (2a + 5b)} & \mathbf{X}
\end{array}
$$

$$
\begin{array}{rcrcrcrcrl}
y & = & at^3 & + & bt^2 & + & ct & & & \ldots \times \mathbf{0} \\
y' & = & & & 3at^2 & + & 2bt & + & c & \ldots \times \mathbf{5} \\
y'' & = & & & & + & 6at & + & 2b & \ldots \times \mathbf{1} \\
\hline
6t^2 - t - 1 & = & \multicolumn{7}{l}{15at^2 + (6a + 10b)t + (2b + 5c)} &
\end{array}
$$

$$\Rightarrow \quad y_p = \frac{2}{5}t^3 - \frac{17}{50}t^2 - \frac{8}{125}t \quad \mathbf{O}$$

Differential Equations with a Variable Term

ex: $y'' + 3y' - 4y = 2e^{-4t} \quad \Rightarrow \quad y_p$**?**

$$\begin{aligned} y &= Be^{-4t} & \ldots \times -4 \\ y' &= -4Be^{-4t} & \ldots \times 3 \\ y'' &= 16Be^{-4t} & \ldots \times 1 \\ \hline 2e^{-4t} &= 0 \quad \textbf{X} & \end{aligned}$$

$$\begin{aligned} y &= Bte^{-4t} & \ldots \times -4 \\ y' &= (1-4t)Be^{-4t} & \ldots \times 3 \\ y'' &= (-8+16t)Be^{-4t} & \ldots \times 1 \\ \hline 2e^{-4t} &= -5Be^{-4t} \quad \Rightarrow & y_p = \frac{-2}{5}te^{-4t} \quad \textbf{O} \end{aligned}$$

Differential Equations with a Variable Term

ex: $y'' + y' + 3y = \sin t \quad \Rightarrow \quad y_p$**?**

$$\begin{array}{rcll} y & = & A_1 \sin t + A_2 \cos t & \ldots \times \mathbf{3} \\ y' & = & -A_2 \sin t + A_1 \cos t & \ldots \times \mathbf{1} \\ y'' & = & -A_1 \sin t - A_2 \cos t & \ldots \times \mathbf{1} \\ \hline \sin t & = & (2A_1 - A_2) \sin t + (A_1 + 2A_2) \cos t & \end{array}$$

$$\Rightarrow \quad y_p = \frac{2}{5} \sin t - \frac{1}{5} \cos t$$ **O**

Higher Order Linear Differential Equations

$$y^{(n)}(t) + a_1 y^{(n-1)}(t) + \cdots + a_{n-1}y' + a_n y = b$$

$$\Rightarrow \; r^n + a_1 r^{n-1} + \cdots + a_{n-1} r + a_n = 0 \quad \Rightarrow \quad r_1, r_2, \cdots r_n$$

- distinct real roots: $\sum_i A_i e^{r_i t}$
- repeated real roots: $\sum_j A_j t^j e^{rt}$
- conjugate complex roots: $e^{\alpha t}(A \cos \beta t + B \sin \beta t)$
- repeated complex roots: $\sum_k t^k e^{\alpha t}(A_k \cos \beta t + B_k \sin \beta t)$

Higher Order Linear Differential Equations

ex: $y^{(4)} + 6y''' + 14y'' + 16y' + 8y = 24$

$\Rightarrow\ r^4 + 6r^3 + 14r^2 + 16r + 8 = 0$

$(r+2)^2(r^2+2r+2) = 0 \quad \Rightarrow \quad r = -2,\ -2,\ -1 \pm i$

$\Rightarrow\ y(t) = A_1 e^{-2t} + A_2 t e^{-2t} + e^{-t}(A_3 \cos t + A_4 \sin t) + 3$

ex: $(2r+3)^3(r-2)(r^2+r+1)^2 = 0$

$$\begin{aligned} y_c &= A_1 e^{-1.5t} + A_2 t e^{-1.5t} + A_3 t^2 e^{-1.5t} + A_4 e^{2t} \\ &+ e^{-1/2t}[A_5 \cos(\sqrt{3}/2)t + A_6 \sin(\sqrt{3}/2)t] \\ &+ e^{-1/2t} t [A_7 \cos(\sqrt{3}/2)t + A_8 \sin(\sqrt{3}/2)t] \end{aligned}$$

Convergence and the Routh Theorem

- The **real parts** of **all** of the roots of the nth-degree polynomial equation

$$a_0 r^n + a_1 r^{n-1} + \cdots + a_{n-1} r + a_n = 0$$

are **negative if and only if** the first n of the following sequence of determinants

$$|a_1|; \quad \begin{vmatrix} a_1 & a_3 \\ a_0 & a_2 \end{vmatrix}; \quad \begin{vmatrix} a_1 & a_3 & a_5 \\ a_0 & a_2 & a_4 \\ 0 & a_1 & a_3 \end{vmatrix};$$

$$\begin{vmatrix} a_1 & a_3 & a_5 & a_7 \\ a_0 & a_2 & a_4 & a_6 \\ 0 & a_1 & a_3 & a_5 \\ 0 & a_0 & a_2 & a_4 \end{vmatrix}; \cdots$$ all are **positive**.

Convergence and the Routh Theorem

ex: $\underset{a_0}{r^4} + \underset{a_1}{6r^3} + \underset{a_2}{14r^2} + \underset{a_3}{16r} + \underset{a_4}{8} = 0$

$$\Rightarrow \quad |6| = 6; \quad \begin{vmatrix} 6 & 16 \\ 1 & 14 \end{vmatrix} = 68; \quad \begin{vmatrix} 6 & 16 & 0 \\ 1 & 14 & 8 \\ 0 & 6 & 16 \end{vmatrix} = 800;$$

$$\begin{vmatrix} 6 & 16 & 0 & 0 \\ 1 & 14 & 8 & 0 \\ 0 & 6 & 16 & 0 \\ 0 & 1 & 14 & 8 \end{vmatrix} = 6,400;$$

$\Rightarrow$ The real parts of all of the roots are negative! (stable)

Convergence and the Routh Theorem

ex:

$$\underset{a_0}{8r^8}+\underset{a_1}{36r^7}+\underset{a_2}{46r^6}-\underset{a_3}{41r^5}-\underset{a_4}{222r^4}-\underset{a_5}{367r^3}-\underset{a_6}{342r^2}-\underset{a_7}{189r}-\underset{a_8}{54}=0$$

$$\Rightarrow \quad |36|=36; \quad \begin{vmatrix} 36 & -41 \\ 8 & 46 \end{vmatrix} = 1,984; \quad \begin{vmatrix} 36 & -41 & -367 \\ 8 & 46 & -222 \\ 0 & 36 & -41 \end{vmatrix} = 100,672;$$

$$\begin{vmatrix} 36 & -41 & -367 & -189 \\ 8 & 46 & -222 & -342 \\ 0 & 36 & -41 & -367 \\ 0 & 8 & 46 & -222 \end{vmatrix} = 4,561,920; \ \ldots$$

Difference Equations

refer to textbook

Ch.17 Discrete Time: First-Order Difference Equations

Ch.18 Higher-Order Difference Equations

First-Order Difference Equations

- $\triangle\, y_t \equiv y_{t+1} - y_t$ **ex:** $\triangle\, y_t = 2$

$\Rightarrow \quad y_{t+1} - y_t = 2$ or $y_{t+1} = y_t + 2$

Iterative Method

$$\begin{aligned}
y_1 &= y_0 + 2 \\
y_2 &= y_1 + 2 = (y_0 + 2) + 2 = y_0 + 2(2) \\
y_3 &= y_2 + 2 = (y_0 + 2(2)) + 2 = y_0 + 3(2) \\
&\vdots \\
y_t &= y_0 + t(2) = y_0 + 2t
\end{aligned}$$

First-Order Difference Equations

ex: $\triangle\, y_t = -0.1 y_t \quad \Rightarrow \quad y_{t+1} = 0.9 y_t$

Iterative Method

$$\begin{aligned}
y_1 &= 0.9 y_0 \\
y_2 &= 0.9 y_1 = (0.9)^2 y_0 \\
y_3 &= 0.9 y_2 = (0.9)^3 y_0 \\
&\vdots \\
y_t &= (0.9)^t y_0
\end{aligned}$$

First-Order Difference Equations

- $y_{t+1} + ay_t = c$

complete equation: $y_{t+1} + ay_t = c$

Try $y_t = k \quad \Rightarrow \quad y_p = \frac{c}{1+a} \; (a \neq -1)$

reduced equation: $y_{t+1} + ay_t = 0$

$y_t = Ab^t \quad \Rightarrow \quad y_c = A(-a)^t$

$$\Rightarrow \; y_t = A(-a)^t + \frac{c}{1+a} = \left[y_0 - \frac{c}{1+a}\right](-a)^t + \frac{c}{1+a}$$

ex: $y_{t+1} - 5y_t = 1$

$$\Rightarrow \; y_t = A(5)^t - \frac{1}{4} = (y_0 + \frac{1}{4}) \cdot 5^t - \frac{1}{4}$$

The Cobweb Model

- Consider a situation in which the producer's output decision must be made one period in advance of the actual date.

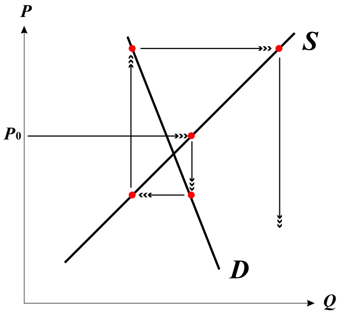
$$\Rightarrow \quad Q_{dt} = \alpha - \beta P_t \qquad (\alpha, \beta > 0)$$
$$Q_{st} = -\gamma + \delta P_{t-1} \qquad (\gamma, \delta > 0)$$

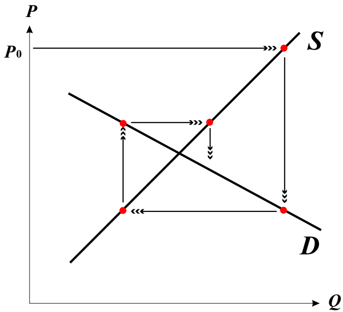
$$\Rightarrow \quad \beta P_t + \delta P_{t-1} = \alpha + \gamma \quad \text{or} \quad P_{t+1} + \frac{\delta}{\beta} P_t = \frac{\alpha + \gamma}{\beta}$$

$$\Rightarrow \quad P_t = (P_0 - \frac{\alpha + \gamma}{\beta + \delta})(\frac{-\delta}{\beta})^t + \frac{\alpha + \gamma}{\beta + \delta}$$

explosive $\qquad >$
$\Rightarrow$ uniform oscillation if $\delta = \beta$
damped $\qquad <$

The Cobweb Model





2nd-Order Difference Equations

$y_{t+2} + a_1 y_{t+1} + a_2 y_t = c$ complete equation

1. Look for y_p

ex: $y_{t+2} - 3y_{t+1} + 4y_t = 6$ try $\overrightarrow{y_t} = k$ $\Longrightarrow$ $y_p = 3$ **O**

ex: $y_{t+2} + y_{t+1} - 2y_t = 12$ try $y_t = k$ $\Longrightarrow$ $0 = 12$ **X**

try $y_t = kt$ $\Longrightarrow$ $y_p = 4t$ **O**

ex: $y_{t+2} - 2y_{t+1} + y_t = 5$

2nd-Order Difference Equations

2. Solve y_c

$$y_{t+2} + a_1 y_{t+1} + a_2 y_t = 0 \quad \text{reduced equation}$$

- Let $y_t = Ab^t$, so that $y_{t+2} = Ab^{t+2}$ and $y_{t+1} = Ab^{t+1}$

$\Rightarrow Ab^t(b^2 + a_1 b + a_2) = 0$, we call $b^2 + a_1 b + a_2 = 0$ as a characteristic (or auxiliary) equation. (**Can** $A = 0$ **happen?)**

$$\Rightarrow b_1,\ b_2 = \frac{-a_1 \pm \sqrt{{a_1}^2 - 4a_2}}{2} \quad \Rightarrow \quad y_1 = A_1 {b_1}^t, \quad y_2 = A_2 {b_2}^t$$

$$\Rightarrow y_c = y_1 + y_2 = A_1 {b_1}^t + A_2 {b_2}^t$$

2nd-Order Difference Equations

- Case 1. Two distinct real roots ($a_1^2 > 4a_2$)

ex: $y_{t+2} + y_{t+1} - 2y_t = 12$

$$\Rightarrow \; b^2 + b - 2 = (b+2)(b-1) = 0 \quad \Rightarrow \quad b_1 = 1, \; b_2 = -2$$

$$\Rightarrow \; y_t = y_c + y_p = A_1(1)^t + A_2(-2)^t + 4t$$

If we let $y_0 = 4$ and $y_1 = 5$, then

$A_1 + A_2 = 4$ and $A_1 - 2A_2 + 4 = 5$

$$\Rightarrow \; A_1 = 3, \; A_2 = 1, \text{ and } \; y_t = 3 + (-2)^t + 4t$$

- Case 2. Two repeated real roots ($a_1^2 = 4a_2 \quad \Rightarrow \quad b = -\frac{a_1}{2}$)

ex: $y_{t+2} + 6y_{t+1} + 9y_t = 4$

$\Rightarrow b^2 + 6b + 9 = (b+3)^2 = 0 \qquad \Rightarrow \qquad b_1 = b_2 = -3$

$\Rightarrow y_c = A_1(-3)^t + A_2(-3)^t = A_3(-3)^t$

(**Only one constant can be identified!**)

If we let $y_c = A_4 t b^t$ (**Can it be another solution?**)

then $y_t = (A_3 + A_4 t)(-3)^t + \dfrac{1}{4}$

- Case 3. Two (conjugate) complex roots ($a_1^2 < 4a_2$)

$$b_1,\ b_2 = \frac{-a_1 \pm \sqrt{4a_2 - {a_1}^2}\ i}{2} = \alpha \pm \beta i$$

$$\begin{aligned} y_c &= A_1(\alpha + \beta i)^t + A_2(\alpha - \beta i)^t \\ &= A_1 R^t(\cos\theta t + i\sin\theta t) + A_2 R^t(\cos\theta t - i\sin\theta t) \\ &= R^t\left[(A_1 + A_2)\cos\theta t + (A_1 - A_2)i\sin\theta t)\right] \\ &= R^t(A_5\cos\theta t + A_6\sin\theta t) \end{aligned}$$

- Case 3. Two (conjugate) complex roots ($a_1^2 < 4a_2$)

ex: $y_{t+2} + \frac{1}{4}y_t = 5 \quad \Rightarrow \quad y_p = 4$

$$b^2 + \frac{1}{4} = 0 \quad \Rightarrow \quad b = \pm\frac{1}{2}i = \frac{1}{2}(\cos\frac{\pi}{2} \pm i\sin\frac{\pi}{2})$$

$$\Rightarrow \; y_t = (\frac{1}{2})^t(A_5\cos\frac{\pi}{2}t + A_6\sin\frac{\pi}{2}t) + 4$$

ex: $y_{t+2} - 4y_{t+1} + 16y_t = 0 \quad \Rightarrow \quad y_p = 0$

$$b^2 - 4b + 16 = 0 \quad \Rightarrow \quad b = 2 \pm 2\sqrt{3}i = 4(\cos\frac{\pi}{3} \pm i\sin\frac{\pi}{3})$$

$$\Rightarrow \; y_t = 4^t(A_5\cos\frac{\pi}{3}t + A_6\sin\frac{\pi}{3}t)$$

The Convergence of the Time Path

- Case 1. Two distinct real roots

  $y_c = A_1 {b_1}^t + A_2 {b_2}^t$

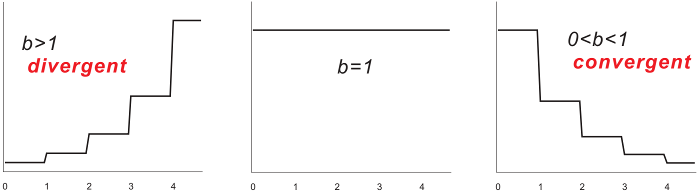
- Case 2. Two repeated real roots

  $y_c = (A_3 + A_4 t) b^t$

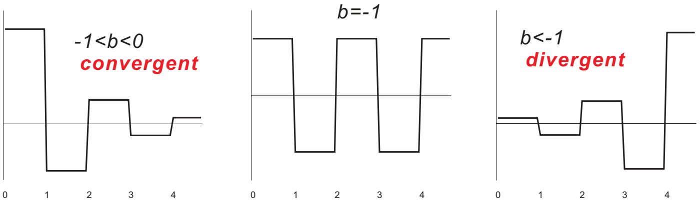
- Case 3. Two (conjugate) complex roots

  $y_c = R^t (A_5 \cos \theta t + A_6 \sin \theta t)$

nonoscillatory



oscillatory



Difference Equations with a Variable Term

ex: $y_{t+2} + y_{t+1} - 3y_t = 7^t \quad \Rightarrow \quad y_p$**?**

$$
\begin{array}{rcll}
y_t & = & B(7^t) & \ldots \times -3 \\
y_{t+1} & = & B(7^{t+1}) = 7B(7^t) & \ldots \times 1 \\
y_{t+2} & = & B(7^{t+2}) = 49B(7^t) & \ldots \times 1 \\
\hline
7^t & = & 53B(7^t) &
\end{array}
$$

$\Rightarrow B = \frac{1}{53} \quad \Rightarrow \quad y_p = \frac{1}{53}7^t$ **O**

Difference Equations with a Variable Term

ex: $y_{t+2} - 5y_{t+1} - 6y_t = 2 \cdot 6^t \quad \Rightarrow \quad y_p$**?**

$$\begin{aligned} y_t &= B(6^t) & \ldots \times -6 \\ y_{t+1} &= B(6^{t+1}) = 6B(6^t) & \ldots \times -5 \\ y_{t+2} &= B(6^{t+2}) = 36B(6^t) & \ldots \times 1 \\ \hline 2 \cdot 6^t &= 0 \quad \text{X} \end{aligned}$$

$$\begin{aligned} y_t &= Bt(6^t) & \ldots \times -6 \\ y_{t+1} &= B(t+1)(6^{t+1}) = 6B(t+1)(6^t) & \ldots \times -5 \\ y_{t+2} &= B(t+2)(6^{t+2}) = 36B(t+2)(6^t) & \ldots \times 1 \\ \hline 2 \cdot 6^t &= 42B(6^t) \quad \Rightarrow \quad y_p = \frac{1}{21}t(6^t) \quad \text{O} \end{aligned}$$

Difference Equations with a Variable Term

ex: $y_{t+2} + 5y_{t+1} + 2y_t = t^2 \quad \Rightarrow \quad y_p$**?**

$$\begin{aligned} y_t &= at^2 + bt + c & \ldots \times \mathbf{2} \\ y_{t+1} &= a(t+1)^2 + b(t+1) + c \\ &= at^2 + (2a+b)t + (a+b+c) & \ldots \times \mathbf{5} \\ y_{t+2} &= a(t+2)^2 + b(t+2) + c \\ &= at^2 + (4a+b)t + (4a+2b+c) & \ldots \times \mathbf{1} \\ \hline t^2 &= 8at^2 + (14a+8b)t + (9a+7b+8c) \end{aligned}$$

$$\Rightarrow a = \frac{1}{8},\ b = \frac{-7}{32},\ c = \frac{13}{256} \quad \Rightarrow \quad y_p = \frac{1}{8}t^2 - \frac{7}{32}t + \frac{13}{256}$$ **O**

ex: $y_{t+2} + 5y_{t+1} + 2y_t = 3^t + 2t + 4t^2$

Higher Order Linear Difference Equations

$$y_{t+n} + a_1 y_{t+n-1} + \cdots + a_{n-1} y_{t+1} + a_n y_t = b$$

$$\Rightarrow \; b^n + a_1 b^{n-1} + \cdots + a_{n-1} b + a_n = 0 \quad \Rightarrow \quad b_1, b_2, \cdots b_n$$

- distinct real roots: $\sum_i A_i {b_i}^t$
- repeated real roots: $\sum_j A_j t^j b^t$
- conjugate complex roots: $R^t(A\cos\theta t + B\sin\theta t)$
- repeated complex roots: $\sum_k t^k R^t(A_k\cos\theta t + B_k\sin\theta t)$

Higher Order Linear Difference Equations

ex: $y_{t+3} - \frac{7}{8}y_{t+2} + \frac{1}{8}y_{t+1} + \frac{1}{32}y_t = 9$

$\Rightarrow\ b^3 - \frac{7}{8}b^2 + \frac{1}{8}b + \frac{1}{32} = 0$

$(2b-1)^2(8b+1) = 0 \quad \Rightarrow \quad b = \frac{1}{2},\ \frac{1}{2},\ -\frac{1}{8}$

$\Rightarrow\ y_t = A_1(\frac{1}{2})^t + A_2 t(\frac{1}{2})^t + A_3(\frac{-1}{8})^t + 32$

ex: $y_{t+4} + 6y_{t+3} + 14y_{t+2} + 16y_{t+1} + 8y_t = 24$

$\Rightarrow\ b^4 + 6b^3 + 14b^2 + 16b + 8 = 0$

$(b+2)^2(b^2+2b+2) = 0 \quad \Rightarrow \quad b = -2,\ -2,\ -1 \pm i$

$\Rightarrow\ y_t = A_1(-2)^t + A_2 t(-2)^t + (\sqrt{2})^t(A_3 \cos\frac{3\pi}{4}t + A_4 \sin\frac{3\pi}{4}t) + \frac{8}{15}$

Convergence and the Schur Theorem

- The roots of the nth-degree polynomial equation

$$a_0 b^n + a_1 b^{n-1} + \cdots + a_{n-1} b + a_n = 0$$

will be **less than unity in absolute value** **if and only if** the following n determinants

$$\triangle_1 = \begin{vmatrix} a_0 & a_n \\ a_n & a_0 \end{vmatrix}; \qquad \triangle_2 = \begin{vmatrix} a_0 & 0 & a_n & a_{n-1} \\ a_1 & a_0 & 0 & a_n \\ a_n & 0 & a_0 & a_1 \\ a_{n-1} & a_n & 0 & a_0 \end{vmatrix}; \qquad \cdots$$

$$\triangle_n = \begin{vmatrix} a_0 & 0 & \cdots & 0 & a_n & a_{n-1} & \cdots & a_1 \\ a_1 & a_0 & \cdots & 0 & 0 & a_n & \cdots & a_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n-1} & a_{n-2} & \cdots & a_0 & 0 & 0 & \cdots & a_n \\ a_n & 0 & \cdots & 0 & a_0 & a_1 & \cdots & a_{n-1} \\ a_{n-1} & a_n & \cdots & 0 & 0 & a_0 & \cdots & a_{n-2} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ a_1 & a_2 & \cdots & a_n & 0 & 0 & \cdots & a_0 \end{vmatrix}$$

all are **positive**.

Convergence and the Schur Theorem

ex: $\underset{a_0}{b^2} + \underset{a_1}{3b} + \underset{a_2}{2} = 0$

$$\Rightarrow \quad \triangle_1 = \begin{vmatrix} 1 & 2 \\ 2 & 1 \end{vmatrix} = -3; \qquad \triangle_2 = \begin{vmatrix} 1 & 0 & 2 & 3 \\ 3 & 1 & 0 & 2 \\ 2 & 0 & 1 & 3 \\ 3 & 2 & 0 & 1 \end{vmatrix} \quad \Rightarrow \quad \text{divergent!}$$

ex: $\underset{a_0}{6b^2} + \underset{a_1}{b} - \underset{a_2}{1} = 0$

$$\Rightarrow \quad \triangle_1 = \begin{vmatrix} 6 & -1 \\ -1 & 6 \end{vmatrix} = 35; \qquad \triangle_2 = \begin{vmatrix} 6 & 0 & -1 & 1 \\ 1 & 6 & 0 & -1 \\ -1 & 0 & 6 & 1 \\ 1 & -1 & 0 & 6 \end{vmatrix} = 1176$$

$\Rightarrow$ All roots are less than unity in absolute value! (convergent)

Simultaneous Equations

refer to textbook

Ch.19 Simultaneous Differential and Difference Equations

Transformation of a Higher-Order Dynamic Equation

ex:

$$y_{t+3} + a_1 y_{t+2} + a_2 y_{t+1} + a_3 y_t = c$$

$$\begin{cases} z_{t+1} & & & +a_1 z_t & +a_2 x_t & +a_3 y_t & = c \\ & x_{t+1} & & -z_t & & & = 0 \\ & & y_{t+1} & & -x_t & & = 0 \end{cases}$$

ex:

$$y^{(3)}(t) + a_1 y''(t) + a_2 y'(t) + a_3 y(t) = c$$

$$\begin{cases} z'(t) & +a_1 x'(t) & & & +a_2 x(t) & +a_3 y(t) & = c \\ & x'(t) & & -z(t) & & & = 0 \\ & & y'(t) & & -x(t) & & = 0 \end{cases}$$

Simultaneous Difference Equations

ex:
$$\begin{aligned} x_{t+1} \qquad\quad &+ 6x_t + 9y_t = 4 \\ y_{t+1} - x_t \qquad &= 0 \end{aligned}$$

$$\Rightarrow \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_{t+1} \\ y_{t+1} \end{bmatrix} + \begin{bmatrix} 6 & 9 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} x_t \\ y_t \end{bmatrix} = \begin{bmatrix} 4 \\ 0 \end{bmatrix}$$

1. **Guess the particular integrals**: x_p and y_p (**Try constants**)

$$\Rightarrow \begin{bmatrix} x_p \\ y_p \end{bmatrix} = \begin{bmatrix} 7 & 9 \\ -1 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 4 \\ 0 \end{bmatrix}$$

$$= \frac{1}{16} \begin{bmatrix} 1 & -9 \\ 1 & 7 \end{bmatrix} \begin{bmatrix} 4 \\ 0 \end{bmatrix} = \begin{bmatrix} 0.25 \\ 0.25 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_{t+1} \\ y_{t+1} \end{bmatrix} + \begin{bmatrix} 6 & 9 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} x_t \\ y_t \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

2. **Solve the complementary functions**: x_c and y_c

$\Rightarrow$ Let $x_t = mb^t$ and $y_t = nb^t$

$$\Rightarrow \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} mb^{t+1} \\ nb^{t+1} \end{bmatrix} + \begin{bmatrix} 6 & 9 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} mb^t \\ nb^t \end{bmatrix}$$

$$= \left(b \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 6 & 9 \\ -1 & 0 \end{bmatrix} \right) \begin{bmatrix} m \\ n \end{bmatrix} b^t = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} b+6 & 9 \\ -1 & b \end{bmatrix} \begin{bmatrix} m \\ n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \left| \begin{array}{cc} b+6 & 9 \\ -1 & b \end{array} \right| = 0 = b^2 + 6b + 9 = (b+3)^2$$

$$\Rightarrow b_1 = b_2 = -3, \quad \begin{bmatrix} 3 & 9 \\ -1 & -3 \end{bmatrix} \begin{bmatrix} m \\ n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \Rightarrow \quad m : n = -3 : 1$$

$$\Rightarrow \begin{bmatrix} x_c \\ y_c \end{bmatrix} = \begin{bmatrix} -3A_3(-3)^t - 3A_4t(-3)^t \\ A_3(-3)^t + A_4t(-3)^t \end{bmatrix}$$

and $\begin{bmatrix} x_t \\ y_t \end{bmatrix} = \begin{bmatrix} -3A_3(-3)^t - 3A_4t(-3)^t + 0.25 \\ A_3(-3)^t + A_4t(-3)^t + 0.25 \end{bmatrix}$

ex:

$$x_{t+1} \qquad - x_t - 1/3y_t = -1$$

$$x_{t+1} + y_{t+1} \qquad - 1/6y_t = 17/2$$

$$\Rightarrow \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_{t+1} \\ y_{t+1} \end{bmatrix} + \begin{bmatrix} -1 & -1/3 \\ 0 & -1/6 \end{bmatrix} \begin{bmatrix} x_t \\ y_t \end{bmatrix} = \begin{bmatrix} -1 \\ 17/2 \end{bmatrix}$$

1. **Guess the particular integrals**: x_p and y_p (**Try constants**)

$$\Rightarrow \begin{bmatrix} x_p \\ y_p \end{bmatrix} = \begin{bmatrix} 0 & -1/3 \\ 1 & 5/6 \end{bmatrix}^{-1} \begin{bmatrix} -1 \\ 17/2 \end{bmatrix}$$

$$= \frac{1}{1/3} \begin{bmatrix} 5/6 & 1/3 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} -1 \\ 17/2 \end{bmatrix} = \begin{bmatrix} 6 \\ 3 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_{t+1} \\ y_{t+1} \end{bmatrix} + \begin{bmatrix} -1 & -1/3 \\ 0 & -1/6 \end{bmatrix} \begin{bmatrix} x_t \\ y_t \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

2. **Solve the complementary functions**: x_c and y_c

$\Rightarrow$ Let $x_t = mb^t$ and $y_t = nb^t$

$$\Rightarrow \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} mb^{t+1} \\ nb^{t+1} \end{bmatrix} + \begin{bmatrix} -1 & -1/3 \\ 0 & -1/6 \end{bmatrix} \begin{bmatrix} mb^t \\ nb^t \end{bmatrix}$$

$$= \left(b \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} + \begin{bmatrix} -1 & -1/3 \\ 0 & -1/6 \end{bmatrix} \right) \begin{bmatrix} m \\ n \end{bmatrix} b^t = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} b-1 & -1/3 \\ b & b-1/6 \end{bmatrix} \begin{bmatrix} m \\ n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{vmatrix} b-1 & -1/3 \\ b & b-1/6 \end{vmatrix} = 0 = b^2 - \frac{5}{6}b + \frac{1}{6} = (b-\frac{1}{2})(b-\frac{1}{3})$$

$$\Rightarrow b_1 = 1/2, \quad \begin{bmatrix} -1/2 & -1/3 \\ 1/2 & 1/3 \end{bmatrix} \begin{bmatrix} m \\ n \end{bmatrix} = 0 \quad \Rightarrow \quad m_1 : n_1 = 2 : -3$$

$$b_2 = 1/3, \quad \begin{bmatrix} -2/3 & -1/3 \\ 1/3 & 1/6 \end{bmatrix} \begin{bmatrix} m \\ n \end{bmatrix} = 0 \quad \Rightarrow \quad m_2 : n_2 = 1 : -2$$

$$\Rightarrow \begin{bmatrix} x_c \\ y_c \end{bmatrix} = \begin{bmatrix} 2A_1(\frac{1}{2})^t + A_2(\frac{1}{3})^t \\ -3A_1(\frac{1}{2})^t - 2A_2(\frac{1}{3})^t \end{bmatrix}$$

and $$\begin{bmatrix} x_t \\ y_t \end{bmatrix} = \begin{bmatrix} 2A_1(\frac{1}{2})^t + A_2(\frac{1}{3})^t + 6 \\ -3A_1(\frac{1}{2})^t - 2A_2(\frac{1}{3})^t + 3 \end{bmatrix}$$

Simultaneous Differential Equations

ex: $x' + 2y' + 2x + 5y = 77$

$$y' + \ x + 4y = 61$$

$$\Rightarrow \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x' \\ y' \end{bmatrix} + \begin{bmatrix} 2 & 5 \\ 1 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 77 \\ 61 \end{bmatrix}$$

1. **Guess the particular integrals**: x_p and y_p (**Try constants**)

$$\Rightarrow \begin{bmatrix} x_p \\ y_p \end{bmatrix} = \begin{bmatrix} 2 & 5 \\ 1 & 4 \end{bmatrix}^{-1} \begin{bmatrix} 77 \\ 61 \end{bmatrix}$$

$$= \frac{1}{3} \begin{bmatrix} 4 & -5 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 77 \\ 61 \end{bmatrix} = \begin{bmatrix} 1 \\ 15 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x' \\ y' \end{bmatrix} + \begin{bmatrix} 2 & 5 \\ 1 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

2. **Solve the complementary functions**: x_c and y_c

$\Rightarrow$ Let $x(t) = me^{rt}$ and $y(t) = ne^{rt}$

$$\Rightarrow \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} rme^{rt} \\ rne^{rt} \end{bmatrix} + \begin{bmatrix} 2 & 5 \\ 1 & 4 \end{bmatrix} \begin{bmatrix} me^{rt} \\ ne^{rt} \end{bmatrix}$$

$$= \left(r \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 2 & 5 \\ 1 & 4 \end{bmatrix} \right) \begin{bmatrix} m \\ n \end{bmatrix} e^{rt} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} r+2 & 2r+5 \\ 1 & r+4 \end{bmatrix} \begin{bmatrix} m \\ n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{vmatrix} r+2 & 2r+5 \\ 1 & r+4 \end{vmatrix} = 0 = r^2 + 4r + 3 = (r+1)(r+3)$$

$$\Rightarrow r_1 = -1, \quad \begin{bmatrix} 1 & 3 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} m \\ n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \Rightarrow \quad m_1 : n_1 = -3 : 1$$

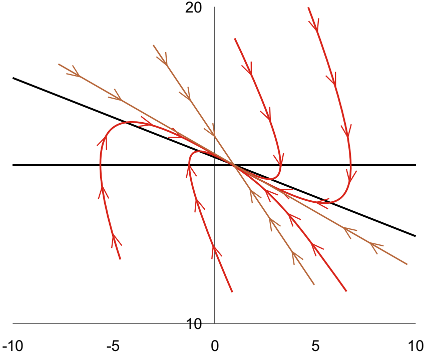
$$r_2 = -3, \quad \begin{bmatrix} -1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} m \\ n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \Rightarrow \quad m_2 : n_2 = -1 : 1$$

$$\Rightarrow \begin{bmatrix} x_c \\ y_c \end{bmatrix} = \begin{bmatrix} -3A_1 e^{-t} - A_2 e^{-3t} \\ A_1 e^{-t} + A_2 e^{-3t} \end{bmatrix}$$

and $$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} -3A_1 e^{-t} - A_2 e^{-3t} + 1 \\ A_1 e^{-t} + A_2 e^{-3t} + 15 \end{bmatrix}$$

Two Variable Phase Diagrams

$$\begin{aligned} x' + 2y' + 2x + 5y &= 77 \\ y' + \ \ x + 4y &= 61 \end{aligned} \quad \Rightarrow \quad \begin{aligned} x' &= 3y - 45 \\ y' &= -x - 4y + 61 \end{aligned}$$



- $(-5, 12)$
- $(5, 20)$
- $A_1 = 0$
- $A_2 = 0$

ex:
$$x' \quad - 2x - y = -4$$
$$y' - 2x + y = 0$$
$$\Rightarrow \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x' \\ y' \end{bmatrix} + \begin{bmatrix} -2 & -1 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -4 \\ 0 \end{bmatrix}$$

1. **Guess the particular integrals**:

$$\Rightarrow \begin{bmatrix} x_p \\ y_p \end{bmatrix} = \begin{bmatrix} -2 & -1 \\ -2 & 1 \end{bmatrix}^{-1} \begin{bmatrix} -4 \\ 0 \end{bmatrix}$$
$$= \frac{1}{-4} \begin{bmatrix} 1 & 1 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} -4 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x' \\ y' \end{bmatrix} + \begin{bmatrix} -2 & -1 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

2. **Solve the complementary functions**:

$$\Rightarrow \left(r \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} -2 & -1 \\ -2 & 1 \end{bmatrix} \right) \begin{bmatrix} m \\ n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} r-2 & -1 \\ -2 & r+1 \end{bmatrix} \begin{bmatrix} m \\ n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{vmatrix} r-2 & -1 \\ -2 & r+1 \end{vmatrix} = 0 = r^2 - r - 4$$

$$\Rightarrow\; r_1 = \frac{1+\sqrt{17}}{2}, \quad \begin{bmatrix} \frac{\sqrt{17}-3}{2} & -1 \\ -2 & \frac{\sqrt{17}+3}{2} \end{bmatrix} \begin{bmatrix} m \\ n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \quad m_1 : n_1 = 2 : (\sqrt{17}-3)$$

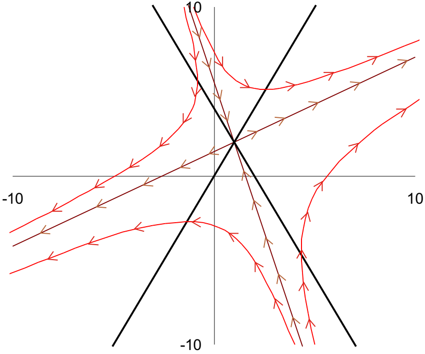
$$r_2 = \frac{1-\sqrt{17}}{2}, \quad \begin{bmatrix} \frac{-\sqrt{17}-3}{2} & -1 \\ -2 & \frac{-\sqrt{17}+3}{2} \end{bmatrix} \begin{bmatrix} m \\ n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \quad m_2 : n_2 = -2 : (\sqrt{17}+3)$$

$$\Rightarrow \begin{bmatrix} x_c \\ y_c \end{bmatrix} = \begin{bmatrix} 2A_1 e^{\frac{1+\sqrt{17}}{2}t} - 2A_2 e^{\frac{1-\sqrt{17}}{2}t} \\ (\sqrt{17}-3)A_1 e^{\frac{1+\sqrt{17}}{2}t} + (\sqrt{17}+3)A_2 e^{\frac{1-\sqrt{17}}{2}t} \end{bmatrix}$$

and $$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} 2A_1 e^{\frac{1+\sqrt{17}}{2}t} - 2A_2 e^{\frac{1-\sqrt{17}}{2}t} + 1 \\ (\sqrt{17}-3)A_1 e^{\frac{1+\sqrt{17}}{2}t} + (\sqrt{17}+3)A_2 e^{\frac{1-\sqrt{17}}{2}t} + 2 \end{bmatrix}$$

$$\begin{aligned} x' \quad &- 2x - y = -4 \\ y' - 2x + y &= 0 \end{aligned} \quad \Rightarrow \quad \begin{aligned} x' &= 2x + y - 4 \\ y' &= 2x - y \end{aligned}$$



- $(5, -10)$
- $(4, -10)$
- $A_1 = 0$
- $A_2 = 0$

ex:
$$x' \quad - x + y = 2$$
$$y' - x - y = 4$$

$$\Rightarrow \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x' \\ y' \end{bmatrix} + \begin{bmatrix} -1 & 1 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$$

1. **Guess the particular integrals**:

$$\Rightarrow \begin{bmatrix} x_p \\ y_p \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ -1 & -1 \end{bmatrix}^{-1} \begin{bmatrix} 2 \\ 4 \end{bmatrix} = \begin{bmatrix} -3 \\ -1 \end{bmatrix}$$

2. **Solve the complementary functions**:

$$\Rightarrow \left(r \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} -1 & 1 \\ -1 & -1 \end{bmatrix} \right) \begin{bmatrix} m \\ n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} r-1 & 1 \\ -1 & r-1 \end{bmatrix} \begin{bmatrix} m \\ n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{vmatrix} r-1 & 1 \\ -1 & r-1 \end{vmatrix} = 0 = r^2 - 2r + 2$$

$$\Rightarrow r_1 = 1+i, \quad \begin{bmatrix} i & 1 \\ -1 & i \end{bmatrix} \begin{bmatrix} m \\ n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \Rightarrow \quad m_1 : n_1 = 1 : -i$$

$$r_2 = 1-i, \quad \begin{bmatrix} -i & 1 \\ -1 & -i \end{bmatrix} \begin{bmatrix} m \\ n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \Rightarrow \quad m_2 : n_2 = 1 : i$$

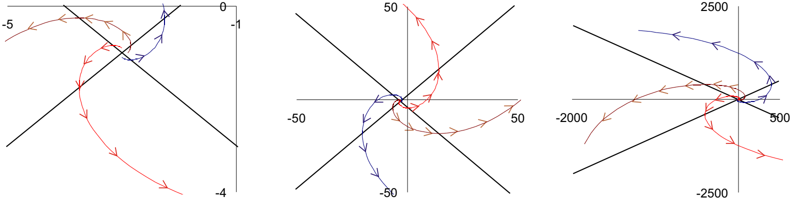
$$\Rightarrow \begin{bmatrix} x_c \\ y_c \end{bmatrix} = \begin{bmatrix} A_1 e^{(1+i)t} + A_2 e^{(1-i)t} \\ -A_1 i e^{(1+i)t} + A_2 i e^{(1-i)t} \end{bmatrix}$$

$$= e^t \begin{bmatrix} A_1(\cos t + i \sin t) + A_2(\cos t - i \sin t) \\ -A_1 i(\cos t + i \sin t) + A_2 i(\cos t - i \sin t) \end{bmatrix}$$

$$= e^t \begin{bmatrix} (A_1 + A_2)\cos t + (A_1 - A_2) i \sin t \\ -(A_1 - A_2) i \cos t + (A_1 + A_2) \sin t \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} e^t(A_5 \cos t + A_6 \sin t) - 3 \\ e^t(-A_6 \cos t + A_5 \sin t) - 1 \end{bmatrix}$$

- $x' = x - y + 2, \qquad y' = x + y + 4$
- $(-2.9, -1)$ • $(-3.1, -1)$



Six Types of Equilibrium

- Given the auxiliary equation $ar^2 + br + c = 0$, one may determine the type of equilibrium with information from
  - the discriminant: $D = b^2 - 4ac$
  - the sum of roots: $r_1 + r_2 = -b/a$
  - the product of roots: $r_1 r_2 = c/a$

$D \geq 0$ real	$D < 0$ conjugate complex
$r_1 + r_2 > 0 \quad r_1 r_2 > 0$ **unstable node**	$r_1 + r_2 > 0$ **unstable focus**
$r_1 + r_2 < 0 \quad r_1 r_2 > 0$ **stable node**	$r_1 + r_2 < 0$ **stable focus**
$r_1 + r_2 \gtreqless 0 \quad r_1 r_2 < 0$ **saddle point**	$r_1 + r_2 = 0$ **vortex**

Linearization of a Nonlinear System

- Given the autonomous system $x' = f(x, y)$ and $y' = g(x, y)$, an equilibrium point $(\overline{x}, \overline{y})$ must satisfy $f(\overline{x}, \overline{y}) = g(\overline{x}, \overline{y}) = 0$.
- The 1st-degree (linear) Taylor expansion around $(\overline{x}, \overline{y})$ gives

$$x' = f(x, y) = f(\overline{x}, \overline{y}) + f_x(\overline{x}, \overline{y})(x - \overline{x}) + f_y(\overline{x}, \overline{y})(y - \overline{y})$$

$$y' = g(x, y) = g(\overline{x}, \overline{y}) + g_x(\overline{x}, \overline{y})(x - \overline{x}) + g_y(\overline{x}, \overline{y})(y - \overline{y})$$

Or

$$x' - f_x(\overline{x}, \overline{y})x - f_y(\overline{x}, \overline{y})y = -f_x(\overline{x}, \overline{y})\overline{x} - f_y(\overline{x}, \overline{y})\overline{y}$$

$$y' - g_x(\overline{x}, \overline{y})x - g_y(\overline{x}, \overline{y})y = -g_x(\overline{x}, \overline{y})\overline{x} - g_y(\overline{x}, \overline{y})\overline{y}$$

$\Rightarrow$ the reduced equations in matrix notation

$$\begin{bmatrix} x' \\ y' \end{bmatrix} - \begin{bmatrix} f_x & f_y \\ g_x & g_y \end{bmatrix}_{(\overline{x}, \overline{y})} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Local Stability Analysis

- The auxiliary equation

$$\begin{vmatrix} r - f_x & -f_y \\ -g_x & r - g_y \end{vmatrix} = r^2 - (f_x + g_y)r + (f_x g_y - f_y g_x) = 0$$

- Denote

$$J_E = \begin{bmatrix} f_x & f_y \\ g_x & g_y \end{bmatrix}_{(\overline{x},\overline{y})}$$

then

$$\begin{aligned} r_1 + r_2 &= \text{tr}(J_E) \\ r_1 r_2 &= \det(J_E) \\ D &= \text{tr}(J_E)^2 - 4 \cdot \det(J_E) \end{aligned}$$

Local Stability Analysis

ex:

$$\begin{aligned} x' &= xy - 2 \\ y' &= 2x - y \end{aligned}$$

ex:

$$\begin{aligned} x' &= x^2 - y \\ y' &= 1 - y \end{aligned}$$

ex:

$$\begin{aligned} x' &= x - y + 2 \\ y' &= x + y + 4 \end{aligned}$$

Optimal Control Theory

$$t = 0 \quad \longrightarrow \quad t = T \text{ or } t = \infty$$

initial time terminal time

- The solution for any control variable:

a single **value** $\longrightarrow$ a complete **time path**

- Define $u(t)$ as a control variable, $y(t)$ as a state variable, and $F(t, y(t), u(t))$ as an instantaneous utility function.

$$\Rightarrow \quad \text{Max} \quad \int_0^T F(t, y, u)\, dt$$

$$\text{s.t.} \quad \dot{y} = f(t, y, u) \qquad + \qquad \text{other conditions}$$

- Terminal Condition:

$$y(T) \exp[-\overline{r}(T) \cdot T] \geq 0$$

where $\overline{r}(t)$ is the average discount rate that between dates 0 and t.

$$\begin{aligned}
\mathcal{L} &= \int_0^T F(t,y,u)\,dt + \int_0^T [\lambda(t)\cdot(f(t,y,u)-\dot{y})]\,dt + \mu\cdot y(T)\exp[-\overline{r}(T)\cdot T] \\
&= \int_0^T [F(t,y,u)+\lambda(t)f(t,y,u)]\,dt - \int_0^T \lambda(t)\,\dot{y}\,dt + \mu\cdot y(T)\exp[-\overline{r}(T)\cdot T]
\end{aligned}$$

integration by parts $\displaystyle\int_0^T \lambda\,dy = \lambda y\Big|_0^T - \int_0^T y\,d\lambda$

$$\begin{aligned}
&= \int_0^T H(t,y,u,\lambda)\,dt + \int_0^T \frac{d\lambda}{dt}y\,dt + \lambda(0)y(0) - \lambda(T)y(T) \\
&\qquad + \mu\cdot y(T)\exp[-\overline{r}(T)\cdot T] \\
&= \int_0^T \left[H(t,y,u,\lambda) + \frac{d\lambda}{dt}y\right]\,dt + \lambda(0)y(0) - \lambda(T)y(T) \\
&\qquad + \mu\cdot y(T)\exp[-\overline{r}(T)\cdot T]
\end{aligned}$$

- Define (Hamiltonian function)

$$H(t, y, u, \lambda) = F(t, y, u) + \lambda(t) f(t, y, u)$$

- Let $\widetilde{u}(t)$ and $\widetilde{y}(t)$ be the optimal time paths for u and y.
- Now, perturbing $\widetilde{u}(t)$ and $\widetilde{y}(t)$ by arbitrary perturbation function $p_1(t)$ and $p_2(t)$, and then get corresponding neighborhood paths:

$$\begin{aligned} u(t) &= \widetilde{u}(t) + \epsilon \cdot p_1(t) \\ y(t) &= \widetilde{y}(t) + \epsilon \cdot p_2(t) \\ y(T) &= \widetilde{y}(T) + \epsilon \cdot p_2(T) \end{aligned}$$

$$\Longrightarrow \quad \left.\frac{\partial \mathcal{L}}{\partial \epsilon}\right|_{\epsilon=0} = 0$$

$$\begin{aligned}
\frac{\partial \mathcal{L}}{\partial \epsilon} &= \frac{\partial}{\partial \epsilon}\left\{\int_0^T \left[H(t,y,u,\lambda) + \frac{d\lambda}{dt}y\right] dt + \Big(\mu \exp[-\overline{r}(T)\cdot T] - \lambda(T)\Big)y(T)\right\} \\
&= \int_0^T \left[\frac{\partial H}{\partial \epsilon} + \frac{d\lambda}{dt}\frac{\partial y}{\partial \epsilon}\right] dt + \Big(\mu \exp[-\overline{r}(T)\cdot T] - \lambda(T)\Big)\frac{\partial y(T)}{\partial \epsilon}
\end{aligned}$$

where $\dfrac{\partial H}{\partial \epsilon} = \dfrac{\partial H}{\partial u}p_1(t) + \dfrac{\partial H}{\partial y}p_2(t)$

$\dfrac{\partial y}{\partial \epsilon} = p_2(t)$ and $\dfrac{\partial y(T)}{\partial \epsilon} = p_2(T)$

$$\begin{aligned}
&= \int_0^T \left[\frac{\partial H}{\partial u}p_1(t) + \left(\frac{\partial H}{\partial y} + \dot{\lambda}\right)p_2(t)\right] dt \\
&\qquad + \Big(\mu \exp[-\overline{r}(T)\cdot T] - \lambda(T)\Big)p_2(T) = 0
\end{aligned}$$

$$\text{Max} \quad \int_0^T F(t,y,u)\,dt$$

$$\text{s.t.} \quad \dot{y} = f(t,y,u) \qquad + \qquad \text{other conditions}$$

$$\Longrightarrow \qquad H(t,y,u,\lambda) = F(t,y,u) + \lambda(t) f(t,y,u)$$

(1) Pontryagin's maximum principle

$$\frac{\partial H}{\partial u} = 0 \quad \text{or} \quad H(t,y,u^*,\lambda) \geq H(t,y,u,\lambda)$$

(2) state equation

$$\dot{y} = \frac{\partial H}{\partial \lambda} = f(t,y,u)$$

(3) costate equation

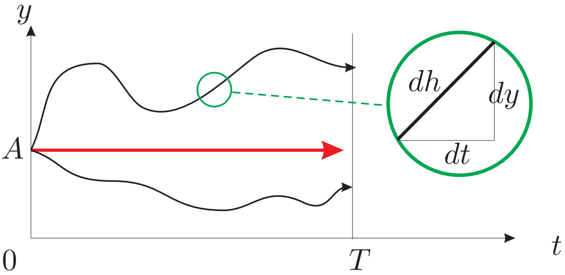
$$\dot{\lambda} = -\frac{\partial H}{\partial y}$$

(4) transversality condition

$$\lambda(T) \geq 0$$

Example 1

Find the shortest distance.



Example 2

$$\text{Max} \quad \int_0^1 (y - u^2)\, dt$$

$$\text{s.t.} \quad \dot{y} = u, \qquad y(0) = 5, \qquad y(1) \text{ free}$$

Example 3

$$\text{Max} \quad \int_0^2 (2y - 3u)\, dt$$

$$\text{s.t.} \quad \dot{y} = y + u, \qquad y(0) = 4, \qquad y(2) \text{ free}, \qquad u(t) \in [0, 2]$$

Neoclassical Optimal Growth Model

$$
\begin{aligned}
Y &= Y(K,L) \text{ is a CRTS production function,} \\
& \quad Y_L > 0, \quad Y_K > 0, \quad Y_{LL} < 0, \quad Y_{KK} < 0 \\
\dot{K} &= I - \delta K, \\
I &= S = Y - C \\
&\Rightarrow \dot{k} = y - c - (n+\delta)k = \phi(k) - c - (n+\delta)k
\end{aligned}
$$

$U(c)$ denotes the social welfare function

$$U'(c) > 0, \quad U''(c) < 0, \quad \lim_{c\to 0} U'(c) = \infty, \quad \lim_{c\to\infty} U'(c) = 0$$

$$\Rightarrow \quad V = \int_0^\infty U(c)e^{-\rho t} L_0 e^{nt}\, dt = \int_0^\infty U(c) e^{-(\rho-n)t}\, dt$$

$$\text{Max} \quad \int_0^\infty U(c)e^{-(\rho-n)t}\,dt$$

$$\text{s.t.} \quad \dot{k} = \phi(k) - c - (n+\delta)k$$

$$\text{and} \quad k(0) = k_0, \qquad 0 \le c(t) \le \phi(k)$$

$$\Rightarrow \quad H = U(c)e^{-(\rho-n)t} + \lambda\Big[\phi(k) - c - (n+\delta)k\Big]$$

(1) $\dfrac{\partial H}{\partial c} = U'(c)e^{-(\rho-n)t} - \lambda = 0$

(2) $\dot{k} = \dfrac{\partial H}{\partial \lambda} = \phi(k) - c - (n+\delta)k$

(3) $\dot{\lambda} = -\dfrac{\partial H}{\partial k} = -\lambda\Big[\phi'(k) - (n+\delta)\Big]$